The Little Known Story of *F* = *ma* and Beyond

Amitabha Ghosh

$F = ma$ is the most frequently used equation in both science and engineering. However, not many are very familiar with the evolutionary process through which the equation finally emerged. This article presents a brief account of the origin of the modern concept of force-matter interaction and the transitory concepts. Towards the end, the article also takes up some still-to-be-resolved matters associated with the equation.

Amitabha Ghosh was a Professor at IIT Kanpur till his retirement in 2006. In between he spent five years (1997 to 2002) at IIT Kharagpur as the Director. Currently he is working as a Senior Scientist of the Indian National Science Academy, New Delhi and is ssociated with both IIT Kanpur and Bengal Engineering & Science University, Sibpur as Honorary Distinguished Professor.

It is doubtless that the most extensively used quantitative law of natural science is $F = ma$. In almost all branches of science and engineering this law plays a pivotal role. For this reason all students are introduced to this law of motion at the school level. However, it is strange that very few users of this law are familiar with the evolutionary process through which the law has been discovered. Most are of the understanding that Newton discovered this law and wrote it in his famous book *Principia* and there is no need to have a relook into the issues involved as all aspects are fully understood.

The primary objective of this article is to present the wonderful evolutionary history behind the development of the profound understanding required to unearth the mystery of the science of motion. It is also important to know about the contributions of great intellectuals of the past whose understandings and findings ultimately led Newton to synthesize their ideas proposing the laws of dynamics. This may also help researchers and students of science to realize how actual progress is made in scientific discoveries through the cumulative accumulation of the contributions by a large number of scientists over a long period of time.

Keywords
Newton's laws, Kepler's laws.

Once humans discovered agriculture, food supply became assured and the wandering nature of their life changed to a more stable character. Thus, for the first time, they became aware of their surroundings and noticed the natural phenomena happening around them. Most likely the first thing that attracted their attention was the movement of objects, i.e., motion. But the bewildering complexity and variety of all types of motions made the understanding of the science behind these phenomena difficult. Aristotle (384–322 BC) was the first philosopher who provided a comprehensive theory of motion. Since the concept of force as a cause of motion was beyond comprehension during those days, all motions which took place without the direct observable intervention of an agent (like hand) were termed as 'natural motions'. This way it was explained that a heavy object falls towards the Earth as the centre of the Earth was the 'natural place' of these objects. Similarly, fire and fiery objects moved upwards as the 'heaven' was their natural place. Those motions which defied the 'natural' expectations were termed as 'violent motions' (like a stone thrown upwards).

The situation remained more or less unchanged for about two millennia until Kepler (1571–1630) started investigating planetary motions. Till that time all philosophers and scientists considered the heavenly objects to be made up of a substance called the 'fifth element' (different from the four basic elements constituting all terrestrial objects) whose natural motion was uniform circular motion around the centre of the universe (till Copernicus's time it was near the centre of the Earth); later this changed to the theory that all planetary motions could be described by systems of perfect circles. However the centres of the circles describing the planets' motions were void points in space a little away from the centre of the Earth (after Copernicus these were a little away from the Sun)! Using the very accurately observed systematic and extensive data, accumulated by his employer Tycho Brahe (1546–1601), Kepler was able to demonstrate that the planets moved in trajectories fixed in the helio-astral space (except for the extremely small precessional motion). The concept of an orbit, i.e., a path fixed in the helio-astral space that is followed by a planet, was first introduced by Kepler. Initially he also considered (like all astronomers before him for about 2000 years) the orbits to be eccentric circles with the Sun a little away from their centres as shown in *Figure* 1.

The line AP, the diameter passing through the centre O and the Sun, is called the 'line of apsides'. For the first time in the history of astronomy, Kepler proved that the orbital planes of all planets were tilted at small angles with the ecliptic plane, i.e., the orbital plane of the Earth. Kepler was acutely uncomfortable with

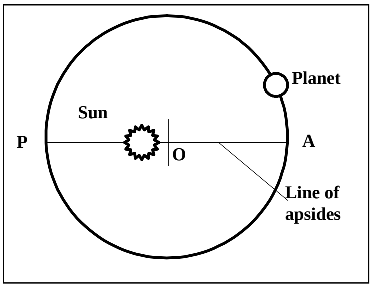


Figure 1. Eccentric circle as planetary orbit.

the idea that a huge object like The Sun just sits idle a little away from the centres, around which the planets move, without doing anything! He intuitively felt that the Sun must be behind the orbital motions of all the planets. His suspicion was intensified when he found that the orbital planes of all planets contain one common object – the Sun and the apsidal lines of the orbits of all planets pass through the Sun! Thus, Kepler arrived at the vague concept of an entity like 'force' that causes bodies to move. His suspicion transformed into a firm conviction that the Sun drives all the planets in their orbits by extending a force through space to these heavenly bodies. *The concept of force in the modern sense was born.*

It was known since antiquity that the motions of the planets along their respective circular orbits were not uniform as seen from the centre of the circle. Ptolemy introduced the concept of 'equant', a point with respect to which a planet describes the circular orbit with uniform angular speed. *Figure* 2 shows the equant for a planet. When a planet goes from a position P_1 to P_2 in a time interval Δt and during an equal time interval the planet goes from position P_3 to P_4, so that the rate of change of angle θ about the equant point E is constant. Kepler, who was obsessed with the idea that the Sun forces the planets to move, came with a physical reason behind the above feature of planetary motion. He reasoned that as a planet at the aphelion position A is farthest from the Sun, the Sun's force on the planet will be minimum resulting in minimum speed of the planet and

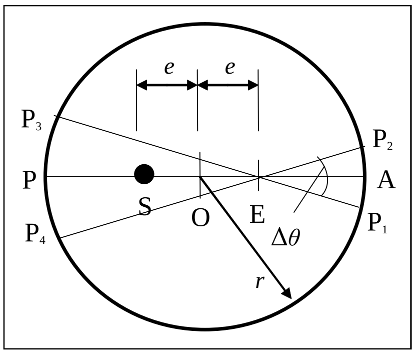


Figure 2. Concept of equant.

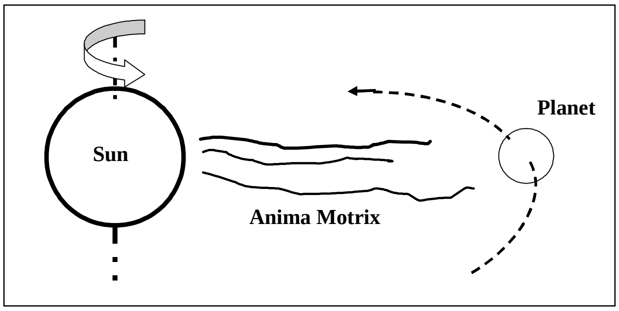


Figure 3. Kepler's idea of gravitational interaction between the Sun and a planet.

the distance travelled in the given time interval, P_1P_2, will be minimum. On the other hand at the perihelion position P the planet is nearest to the Sun and will move at the fastest speed describing the largest arc P_3P_4 in the given time interval Δt. Initially Kepler formulated a 'distance law' according to which the influence of the Sun, i.e., an entity like the force exerted, was inversely proportional to the distance. Kepler reasoned that the purpose of this force emanating from the Sun was to move the planets in the orbital plane and therefore it will spread along the plane making the intensity fall inversely with the distance. He called this force as 'anima motrix' and the scheme is shown in *Figure* 3. Therefore, as per Kepler's theory

$$F \ \alpha \ 1/d$$

As Kepler erroneously assumed speed to be proportional to force in his scheme, the planet's speed was to be inversely proportional to the distance from the Sun. Though he was the first to create a physical model of the planetary motion replacing the 2000-year-old geometric model, because of wrong physics he could not match the observations with his theory.

Kepler also proposed a rate of rotation for the Sun considering the rotational motion of the planet under the influence of the Sun. Galileo's telescopic observation of the sunspots actually revealed the rotation of the Sun and the rate of rotation was found to be close to that predicted by Kepler. Though this provided immense satisfaction to Kepler it was nothing but a pure coincidence! While struggling to explain the observation with his 'physics', Kepler assumed the area law as an approximation for describing the actual motion. He first thought this to be an approximate ad hoc assumption but later he discovered that the area law was an exact law. His analysis and reasoning were as follows:

From *Figure* 4 it is seen that

$$\mathrm{P_1A} = \frac{1}{2}\Delta\theta(r - e),$$

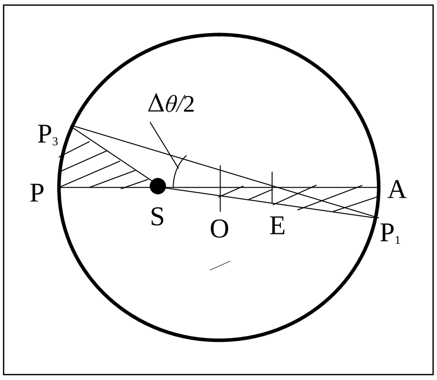


Figure 4. Emergence of area law.

where r is the radius of the circle and e is the distance of the equant point E from the centre O (it should be noted that the distance of the Sun S from the centre O is also e). Again

$$\mathrm{P_3P} = \frac{1}{2}\Delta\theta(r+e).$$

When Δt is very small, the small circular arcs may be approximated as straight lines. The area of $\Delta\mathrm{SP_1A}$ is equal to

$$\frac{1}{2}\mathrm{SA_1.AP_1} = \frac{1}{2}(r+e)\frac{1}{2}\Delta\theta(r-e) = \frac{1}{4}(r^2-e^2)\Delta\theta. \qquad (1)$$

Similarly the area of $\Delta\mathrm{SP_3P}$ is equal to

$$\frac{1}{2}\mathrm{SP_3.P_3P} = \frac{1}{2}(r-e)\frac{1}{2}\Delta\theta(r+e) = \frac{1}{4}(r^2-e^2)\Delta\theta. \qquad (2)$$

Therefore, as $\Delta\theta$ is same for same time intervals, the planet's motion describes equal areas about the Sun in equal times as found from (1) and (2).[1] The reader can easily notice that according to Kepler's 'distance law' the speeds at A and P, V_A and V_P can be expressed as follows with C as the constant:

$$V_\mathrm{A} = C/(r+e); \quad V_\mathrm{P} = C/(r-e).$$

So, the areas of the triangles $\mathrm{SP_1A}$ and $\mathrm{SP_3P}$ for a given time interval Δt can be expressed as follows:

$$\text{Area of } \Delta\mathrm{SP_1A} = \frac{1}{2}\mathrm{SA_1}.\Delta t V_\mathrm{A} = \frac{1}{2}(r+e).C\Delta t/(r+e) = \frac{1}{2}C\Delta t,$$

[1]The reader is advised to prove the area law for a general point on the orbit. This is valid for an elliptic orbit also!

and

$$\text{Area of } \Delta \text{SP}_3\text{P} = \frac{1}{2}\text{SP}.\Delta t V_\text{P} = \frac{1}{2}(r-e).C\Delta t/(r-e) = \frac{1}{2}C\Delta t.$$

This equality is NOT true in general for other locations in the orbit; but this led Kepler along a wrong track for quite some time. However, all these correct and incorrect but apparently correct results convinced Kepler that a force from the Sun is driving the planets.

Subsequently, when Kepler discovered that all planets move in elliptic orbits with the Sun at their foci, there was not a shred of doubt in his mind that the Sun drives the planets. Unfortunately, none of the contemporary scientists like Galileo believed in action-at-a-distance and did not accept the concept of gravitation.

Influenced by Gilbert's book on magnetism Kepler believed in the existence of force that can act at a distance. Furthermore as Gilbert demonstrated that the Earth is a big magnet, Kepler took to the idea of an interaction between the Sun and the planets similar to that found between magnets and proposed a theory that can be said to be the forerunner of Newton's theory of universal gravitation. The following passage from *Astronomia Nova* shows how close Kepler came to the concept of gravitation.

> "Gravity is the propensity between like bodies to unite or come together ... so that the Earth draws the stone to it much rather than the stone seeks the Earth .... If two stones were to be placed anywhere in the world outside the range of influence of a third similar body, then each stone, like two magnetic bodies, would come together at an intermediate point, each stone travelling towards the other a distance proportional to the bulk of the other".

The area law, luckily discovered by Kepler, played the most crucial role in his discovery of the elliptic nature of the planetary orbits. However, the nature of action between force and matter, i.e., through a far more subtle interaction involving instantaneous acceleration was completely beyond him. Furthermore, in the absence of a law of inertia of uniform rectilinear motion and the second law, $F = ma$, it was impossible for him to understand that the orbital motion was composed of two motions – a uniform motion due to inertia and an accelerated motion due to an 'attractive force' by the Sun. Notwithstanding the above shortcoming, Kepler's contributions to the early development of the concept of force and its ability to cause motion of matter are phenomenal. Kepler was also the first to introduce the term 'inertia' in the vocabulary of dynamics. This term was used by him to describe the property of matter to remain at rest unless otherwise impressed upon by a force. But he did not have the understanding that matter tends to remain in uniform motion also unless acted upon by a force.

As acceleration plays the central role in dynamics any further progress could be possible only with a proper understanding of accelerated motion. Some early studies on uniformly accelerated change were conducted by William Heytesbury (1313– 1373) at Merton School, Oxford. More or less contemporarily, further progress was made at Paris school under the leadership of Jean Buridan (1295–1356) followed by Nicholas Oresme (1323–1382). However, the world had to wait till Galileo Galilei (1564–1642) conducted extensive studies on uniformly accelerated motion and free fall. He demonstrated that the distance covered under uniform acceleration increases with the second power of time; he derived the relation

$$s = \frac{1}{2}at^2. \tag{3}$$

Galileo made two other critically important contributions that led to the development of the science of motion and discovering the law $F = ma$. He was the first to propose the law of inertia of motion in a somewhat rudimentary form. According to his understanding if an object is given a motion on the surface of the Earth it will continue to move around the Earth if not stopped. Thus it was something like a 'circular' inertial motion. The other major breakthrough by him was the concept of resolving motions into components. Conversely, he showed the composition of motion and was the first to demonstrate that a projectile motion was a combination of an inertial motion (an approximately uniform rectilinear motion, in the small scale according to Galileo) and a uniformly accelerated free fall. Only the concept of the composition of motions could ultimately lead to the theory of orbital motion of the planets.

Closely following Galileo, Rene Descartes (1596–1650) took up the study of dynamics and was the first to correctly enumerate the first law of motion, i.e., a body continues in a uniform rectilinear motion[2] when not obstructed. This came

[2] It is truly strange that the most fundamental type of motion, i.e., uniform rectilinear motion, that plays the foundational role in the science of motion, is never observed in reality and the philosophers for two millennia were misguided by the uniform circular motion of the heavenly bodies. According to scientists the discovery of the first law was, perhaps, the most difficult and important steps in the history of scientific revolution!

from the observation of the motion of the objects flying out of slings. Motion of water droplets thrown away from the periphery of spinning wheels also led to the realization of first law. This kind of study was very popular during that period to understand the situation relating to a spinning Earth! As Rene Descartes' physics and cosmology involved collision of particles, the concept of 'momentum' as a quantitative measure of motion was proposed by him and he attempted to solve collision problems using this concept with only partial success. According to him momentum was the product of speed and bulk of matter.

His friend and disciple, Christian Huygens (1629–1695) took up the study of collision problems and solved the problem for elastic collisions and framed the concept of the law of conservation of linear momentum correctly. However, Huygens' most revolutionary contribution to dynamics was his discovery of the relation between impressed force and the resulting acceleration of a body. His reasoning and thought process that led to this profound discovery are presented below.

Huygens examined the matter of free fall under three different situations as indicated in *Figure* 5. He reasoned that the 'tug' (tension in modern language) T on the string from which an object is hanging prevents it from accelerating downwards due to free fall. Similarly, when the object is on an incline (*Figure* 5b) the tug T' is less and is proportional to the acceleration the object acquires once the tug vanishes (i.e., the string breaks or the object is released). The situation indicated in *Figure* 5c demonstrates the fact that the tug depends only on the acceleration of the object *immediately* after the release and not on the subsequent motion. Thus, Huygens reasoned that the tug that prevents the body from being accelerated is proportional to the acceleration it prevents. Inversely, a tug (or force in modern language) can cause a body to accelerate and the magnitude of the acceleration will be proportional to the tug. Thus, the most important (and

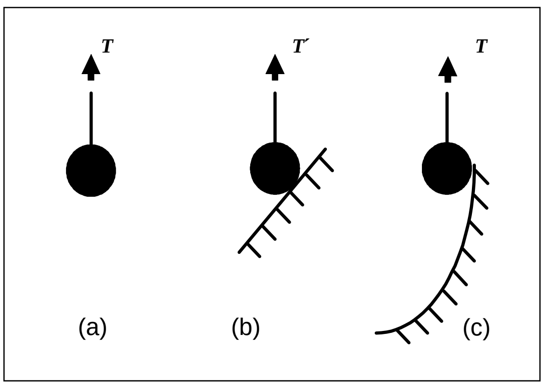


Figure 5. Logical thought experiment by Huygens to discover the proportionality of force with acceleration.

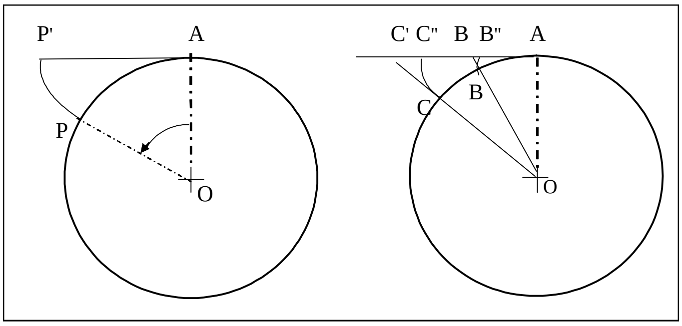


Figure 6. Study of circular motion by Huygens.

difficult too) quantitative relationship in dynamics

$$T \,\alpha\, a \tag{4}$$

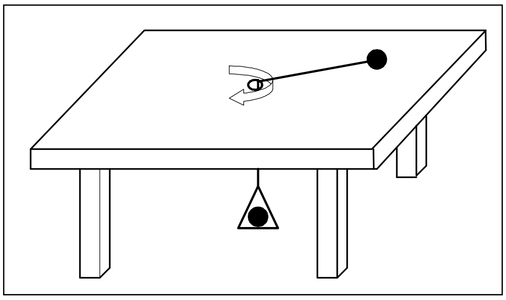
was unraveled by Huygens. He was also the first scientist to arrive at the correct magnitude of centripetal acceleration of an object moving with speed V in a circular path of radius r as V^2/r. He studied the problem in a manner similar to what is presented below. He considered a point on a circle executing uniform rotation. If the particle is freed from the circle at the point A it will move in a straight line according to the law of inertial motion and reach a point P$'$ after a period of time t. Had it remained attached to the circle it would have reached a point P after the same period of time. Huygens first proved that the path of relative motion of the free particle with respect to the particle that is attached to the circle at P is radial at point P (*Figure* 6). With this he demonstrated that the tendency of a free particle is to fly out radially outwards from the point of release. So, a point/particle that is executing uniform circular motion has a relative motion with respect to a free (i.e., unaccelerated) particle that is radially inwards at the point of their separation, i.e., point A. If the circumferential speed be u and the free particle reaches the point B$''$ after a time t (when the particle in the circular motion reaches the point B) then AB$'' = ut$. Furthermore, when t is very small the curve BB$''$ will be indistinguishable from the line BB$'$ (extension of the radial line OB). Now,

$$\mathrm{BB}' = \sqrt{}(\mathrm{OA}^2 + \mathrm{AB}'^2) - \mathrm{OB}.$$

Because OA = OB = r and AB$' \approx$ AB$'' = ut$,

$$\begin{aligned} \mathrm{BB}' &\approx \mathrm{OA}\sqrt{}[1 + (\mathrm{AB}'/\mathrm{OA})^2] - \mathrm{OA} \\ &\approx \frac{1}{2}(u^2/r)t^2. \end{aligned} \tag{5}$$

Figure 7. An experiment on centrifugal force.



Similarly, after a period of time $2t$, the free particle reaches the point C$''$ (very near to C$'$) and the point under circular motion goes to C. The distance is

$$CC' \approx (u^2/r)(2t)^2 . \qquad (6)$$

Thus, the particle undergoing circular motion falls radially inwards with respect to the unaccelerated free particle (at the point of separation) with a uniform acceleration as the distance fallen increases with the second power of time (as observed in free falls by Galileo). The magnitude of this acceleration is given by u^2/r.

Huygens conducted many experiments with slings and through experiments on stabilizing whirling objects on a smooth table by hanging weights (as shown in *Figure* 7) and compared the tug exerted by the weight with the theoretically estimated centripetal acceleration of the object. These results confirmed his intuitive discovery

$$F \ \alpha \ a. \qquad (7)$$

This was the most important discovery in the science of motion and was perfected by Isaac Newton (1642–1727) in the form of his second law of motion. Newton introduced the concept of mass as a measure of quantity of matter and the proportionality was replaced by equality in the following manner

$$F = ma. \qquad (8)$$

The original form of the second law, as proposed by Newton, was, of course, in terms of the rate of change of momentum as we all know. When mass is considered invariant, the rate of change of momentum is nothing but the mass times the acceleration. The reason behind Newton's expressing second law in terms of momentum lies in his work on orbits of planets due to the gravitational attraction of the Sun as indicated in *Figure* 8. He explained the orbital motion as follows. A planet at location A goes in a small time interval Δt to point B.

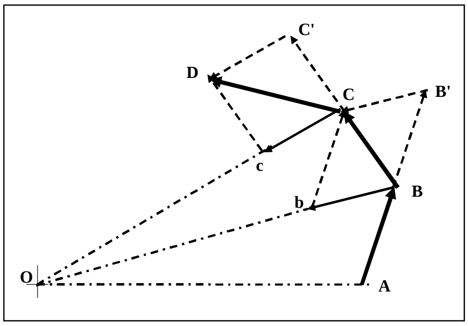


Figure 8. Demonstration of planetary orbit by Newton.

Subsequently, it would have reached position B′ had there been no force acting on it during an equal interval of time Δt in such a way that AB = BB′. But the planet is subjected to an attractive force towards the Sun at O and its effect during the time interval Δt can be represented by an impulse on the planet at B acting along the direction BO. This impulse introduces a corresponding change in the momentum of the planet resulting in a change in the velocity of the planet given by a vector Bb. This causes the planet to arrive at location C during a time interval Δt due to the altered resultant velocity. In a similar way the planet reaches a point D instead of the point C' after a time interval Δt under the influence of the attractive central force due to gravitation. This approach, using the change of momentum due to an impulse during a short time interval, perhaps led Newton to formulate his second law in terms of force and the rate of change of momentum. However, so long as the mass m is treated as an invariant this law is same as $F = ma$.

Contrary to common belief the story of the second law of motion does not end here. Serious questions on the nature of the origin of the 'inertia' property of matter were raised by the contemporary philosophers. The most prominent among them was George Berkeley (1685–1753). He proposed that inertia is not an intrinsic property of matter as suggested by Newton, but it arises out of the interaction of a body with the matter present in the rest of the universe when there is an attempt to accelerate the body. Another major mystery could not be resolved satisfactorily. Though the phenomenon of gravitational interaction between bodies is completely different from the occurrence of a resistive force while accelerating a body, the gravitational mass of a body is always found to be equivalent to another property of the body – its inertial mass. In the 18th century such deep rooted philosophical questions did not have much impact because of the tremendous success of mechanics, as synthesized in the final form by Newton, in explaining all motions and motion-related phenomena. The point was again raised in the second half of the 19th century by the noted Austrian philosopher Ernst Mach

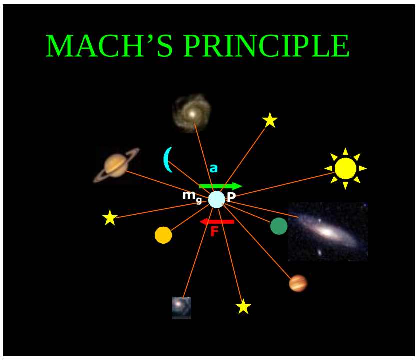


Figure 9. Illustration of Mach's Principle.

(1838–1916), and this time with better success. He suggested that when a body is accelerated all the matter present in the rest of the universe offers a resistance to the acceleration (*Figure* 9), and, that is the origin of inertial force[3]. Since this philosophy received a lot of support from a few scientists (Einstein being one of them) the idea gained a certain degree of legitimacy and became known as 'Mach's Principle'. Of course, it remained just a philosophical statement and for the first time a quantitative modeling of Mach's Principle as attempted by D W Sciama (1926–2001) towards the beginning of the second half of the 20th century. He proposed a model of dynamic gravitational interaction (termed as *inertial induction* by him) according to which the gravitational force between two bodies depends not only on their mutual separation but also on their relative acceleration. In a very crude form the gravitational force on the body A (*Figure* 10) due to the body B can be written as follows:

$$f = -(Gm_{\mathrm{gA}}m_{\mathrm{gB}}/r^2) - (Gm_{\mathrm{gA}}m_{\mathrm{gB}}/c^2r)a, \qquad (9)$$

where r is the distance between the two bodies, a is the acceleration of body A with respect to body B, c is the speed of light, m_{gA} and m_{gB} are the gravitational masses of the two bodies and G represents the gravitational constant. When an object of gravitational mass m accelerates at the rate a with respect to the matter present in the rest of the universe the total resistance experienced by the body can be obtained by summing up the inertial induction with respect to all the matter in the universe. Thus,

$$F = \Sigma f = -\Sigma(Gm_{\mathrm{gB}}/c^2r)ma. \qquad (10)$$

[3] In fact it is not that the faraway matter instantaneously act on the accelerating body to produce the resistance as nothing can move with a speed more than that of light. What is presumed is that all the matter present in the universe generate their influence at the location of the body being accelerated; and this local field interacts with the accelerating body to produce the resistance we term as inertia.

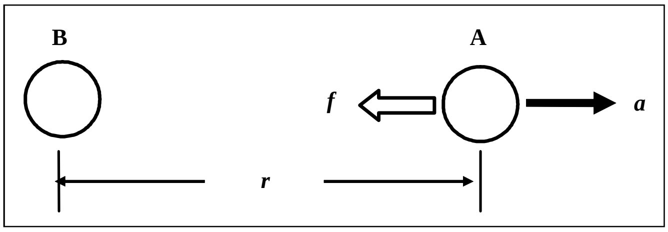


Figure 10. Model of inertial induction.

The sum of the first term in the RHS of (10) becomes zero due to the isotropy and homogeneity of the matter present in the universe when considered in the large scale. The value of the term within the brackets in the RHS of (11) becomes approximately equal to unity resulting in the following relation:

$$F \approx -ma.$$

This shows why the gravitational mass appears as the inertial mass in the force law. It also demonstrates that the second law of motion is not a 'law' but a derived relation from dynamic gravitational interaction of an accelerating body with the matter present in the rest of the universe! However, the mystery still persists as the equivalence is exact and not approximate. How such a fine tuning is possible of the various parameters of the universe to make the equivalence exact and the force law to be $F = ma$ has been also worked out by introducing a velocity-dependent term in the model of inertial induction given by (10), but that is another story.

Suggested Reading

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Address for Correspondence: Amitabha Ghosh, INSA Senior Scientist, Bengal Engineering and Science University, Shibpur, P.O. Botanic Garden, Howrah 711 103, India. Email: amitabha@iitk.ac.in, amitabhaiitk@gmail.com