Snippets of Physics

24. Kepler and his Problem

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The major contribution of Kepler was in discovering his laws of planetary motion, one of which states that planets move in elliptical orbits around the Sun which is located at one of the foci of the ellipse. The motion of a test particle in an inverse square force law is usually called the Kepler problem. We will examine several aspects of this motion in this last installment of Snippets in Physics.

It is a textbook exercise to show that the bound orbits in Newtonian gravity under the influence of an inverse square law force are ellipses. We have already discussed several features of such a motion in previous articles [1, 2]. In particular, this motion has the following features, which are special to this problem and disappear if you change the problem ever so slightly!

• The trajectory of the particle in the velocity space (usually called a hodograph) undergoing such a motion, is a circle [1]. The velocity $\mathbf{v}$ of the particle can be expressed in the form $\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{u}(t)$ where $\mathbf{v}_0$ is a vector constant in magnitude and direction while $\mathbf{u}(t)$ has constant magnitude but varying direction.

• In addition to the angular momentum and energy, which would be conserved in any static central force problem, we have an extra conserved quantity called the Runge–Lenz vector. The magnitude of this vector can be arranged to give the eccentricity of the orbit while its direction is along the major axis of the ellipse. The conservation of this vector ensures that the direction of the major axis does not change and the ellipse stays in place.

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The first casualty when we modify the problem is the Runge–Lenz vector. This means that even when the modification is small, the direction of the ellipse is not protected by a conservation law. This causes the ellipse to precess and it is often of interest to compute the rate of this precession. We shall now discuss how this comes about in different contexts and what it implies.

The study of orbits in external fields is most economically done using the Hamilton–Jacobi equation. Solving the relevant Hamilton–Jacobi equation in the context of a central force problem leads to an action which can be expressed in the form

$$A(t, r; E, L) = -Et + L\theta + S(r; E, L), \tag{1}$$

where (r, θ) are the standard polar coordinates in the plane of orbit, L is the angular momentum, E is the energy and $S(r; E, L)$ has to be determined by integrating the Hamilton–Jacobi equation. The orbital equation $r = r(\theta)$ can be obtained by differentiating A with respect to L and equating it to a constant:

$$\theta + \frac{\partial S}{\partial L} = \theta_0 = \text{constant} . \tag{2}$$

The different contexts we would be interested in only differs in the nature of Hamilton–Jacobi equation; once we obtain the orbital equation in (2) one can compare the different models fairly easily.

Let us start with the standard Newtonian theory. If the particle is moving in a central potential $V(r)$, then from the Hamilton–Jacobi equation $\partial A/\partial t + H = 0$, it is easy to show that S satisfies the equation

$$\left(\frac{\mathrm{d}S}{\mathrm{d}r}\right)^2 = 2m(E - V) - (L^2/r^2). \tag{3}$$

This, in turn, allows us to write the orbital equation in (2) in the form

$$\theta - \theta_0 = \int \frac{\mathrm{d}r(L^2/r^2)}{\left[2m(E - V) - (L^2/r^2)\right]^{1/2}}. \tag{4}$$

Converting this into an equation for $\mathrm{d}r/\mathrm{d}\theta$ and introducing the standard substitution $u \equiv (1/r)$ we can obtain the differential equation satisfied by $u(\theta)$:

$$u'' + u = -\frac{m}{L^2}\frac{\mathrm{d}V}{\mathrm{d}u}, \qquad (5)$$

where the prime denotes differentiation with respect to θ. In the standard Kepler problem, $V = -GMm/r = -GMmu$ so that the right-hand side of (5) becomes a constant and we get the solution $u = \alpha + \beta\cos\theta$ which represents an ellipse.

Let us now ask what happens to the Kepler problem when we introduce physically relevant modifications. The first generalization that you might think of will be to introduce the effects of special relativity. This turns out to be more non-trivial than one might have imagined for the following reason.

In the nonrelativistic context, the motion of a particle under the action of a potential V is governed by the equation $\mathrm{d}p^\alpha/\mathrm{d}t = -\partial_\alpha V$, where $\alpha = 1, 2, 3$ denotes the three spatial components of the momentum $\mathbf{p}$ and ∂_α denotes the derivative with respect to the coordinate x^α. One might have thought that the natural generalization of this Newtonian result into the special relativistic domain would involve the following replacements: Change the three momentum p_α to the four momentum p_i (with $i = 0, 1, 2, 3$), the coordinate time t into the proper time τ of the particle and the three-dimensional gradient ∂_α to the four-gradient ∂_i. This would have led to the equation $\mathrm{d}p_i/\mathrm{d}\tau = -\partial_i V$. Unfortunately, there is a problem with this 'generalization'. The four-velocity u^i satisfies the constraint:

$$u_i u^i = \frac{\mathrm{d}x_i \mathrm{d}x^i}{\mathrm{d}\tau \mathrm{d}\tau} = -\frac{\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = -1 \ , \qquad (6)$$

where we have introduced the summation convention which requires us to sum all repeated indices over the range $i = 0, 1, 2, 3$ and $u_i = \eta_{ij}u^j$ with $\eta_{ij} = dia[-1, 1, 1, 1]$. Since the four-momentum $p^i = mu^i$ is proportional to four-velocity u^i, we have the constraint

$$u^i \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = mu^i \frac{\mathrm{d}u_i}{\mathrm{d}\tau} = \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(u_i u^i) = 0 \ , \qquad (7)$$

where we have used (6). This implies that our potential V has to satisfy the constraints $u^i \partial_i V = 0$; that is, the potential should not change along the world line of the particle which is not possible in general. This is one reason why velocity-independent forces like $\partial_i V$ cannot be introduced in special relativity.

So the generalization to special relativity has to come from some other direction. One possibility is to note that the Kepler problem also arises in electrodynamics when we consider the motion of a test charge in the Coulomb field of another charge. Since we have a fully special relativistic formulation of electrodynamics, we can attempt to study the motion of a particle under the action of a four-vector potential $A^i = (V(r), 0, 0, 0)$ which would correspond to a centrally symmetric electrostatic potential. We can once again write down the Hamilton–Jacobi equation for this case and obtain, in analogy with (3), the differential equation for S given by

$$\left(\frac{\mathrm{d}S}{\mathrm{d}r}\right)^2 = \frac{1}{c^2}(E - V)^2 - \frac{L^2}{r^2} - m^2c^2. \tag{8}$$

It is fairly straightforward to show that, in this case, (5) gets modified to the form

$$u'' + u = -\frac{(E - V)}{L^2c^2}\left(\frac{\mathrm{d}V}{\mathrm{d}u}\right) = -\frac{E/c^2}{L^2}\frac{\mathrm{d}V}{\mathrm{d}u} + \frac{1}{2}\frac{1}{L^2c^2}\frac{\mathrm{d}V^2}{\mathrm{d}u}. \tag{9}$$

Comparing (9) with (5) we see that the first term involves replacement of m by E/c^2 which, of course, makes sense though it brings in a velocity dependence; the second term shows that the potential picks up a V^2 term as a correction which can be traced back to the fact that while $p^2 \propto E$ in the non-relativistic case, $p^2 = (E/c)^2 - m^2c^2$ in special relativity. More formally, we can attempt to define a Newtonian effective potential V_{eff} in which we will obtain the same equation of motion. In the case of motion in a Coulomb field with $V(r) = -\alpha/r = -\alpha u$, this requires us to satisfy the condition

$$\frac{m}{L^2}\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}u} = -\frac{\alpha E}{L^2c^2} - \frac{\alpha^2}{L^2c^2}u \tag{10}$$

which integrates to give

$$V_{\text{eff}} = -\left(\frac{E}{mc^2}\right)\alpha u - \frac{\alpha^2}{mc^2}\frac{u^2}{2}. \tag{11}$$

Since E/mc^2 is γ, we can think of the first term as the original potential transformed to the rest frame of the moving body. The second term is a purely relativistic correction.

In this case of relativistic motion in a Coulomb field, the orbit equation becomes:

$$u'' + \omega^2 u = \frac{\alpha E}{L^2 c^2}; \qquad \omega^2 \equiv 1 - \frac{\alpha^2}{L^2 c^2}, \tag{12}$$

which is again essentially a harmonic oscillator equation. The trajectory obtained by solving (12) can be expressed in the form

$$\frac{1}{r} = \frac{1}{R}\cos(\omega\theta) + \frac{E\alpha}{c^2 L^2 \omega^2}\,, \tag{13}$$

where

$$R \equiv \frac{L\omega^2}{mc}\left[\left(\frac{E}{mc^2}\right)^2 - 1 + \frac{\alpha^2}{c^2 L^2}\right]^{-1/2} \tag{14}$$

is a constant. In a more familiar form, the trajectory is $l/r = (1 + e\cos\omega\theta)$ with

$$l = \frac{c^2 J L^2 \omega^2}{E|\alpha|}; \qquad e^2 = \frac{L^2 c^2}{\alpha^2}\left[1 - \frac{m^2 c^4 \omega^2}{E^2}\right]. \tag{15}$$

It is easy to verify that, when $c \to \infty$, this reduces to the standard equation for an ellipse in the Kepler problem. In terms of the non-relativistic energy $E_{\rm nr} = E - mc^2$, we get, to leading order, $\omega \approx 1, l \approx L^2/m|\alpha|$ and $e^2 \approx 1 + (2E_{\rm nr}L^2/m\alpha^2)$, which are the standard results.

In the fully relativistic case, all these expressions change but the key new effect arises from the fact that $\omega \neq 1$. Due to this factor, the trajectory is not closed and the ellipse precesses. When $\omega \neq 1$, the r in (13) does not return to the value at $\theta = 0$ when $\theta = 2\pi$; instead, we need a further turn by $\Delta\theta$ (the 'angle of precession') for r to return to the original value. This is determined by the condition $(2\pi + \Delta\theta)\omega = 2\pi$. From Eq. (13) we find that the orbit precesses by the angle

$$\Delta\theta = 2\pi\left[\left(1 - \frac{\alpha^2}{c^2 L^2}\right)^{-1/2} - 1\right] \simeq \frac{\pi\alpha^2}{c^2 L^2} \tag{16}$$

per orbit where the second expression is valid for $\alpha^2 \ll c^2 L^2$. This is a purely relativistic effect and vanishes when $c \to \infty$.

There is another peculiar feature that arises in the special relativistic case which has no Newtonian analogue. You would have noticed that ω^2 in (12) has two terms of opposite sign and in obtaining our result in (13), we have tacitly assumed that $\omega^2 > 0$. But in principle, one can have a situation with very low but non-zero angular momentum making $\omega^2 < 0$. This is a feature which the non-relativistic Kepler problem simply does not have and – under such drastic change of circumstances – one can no longer think in terms of precessing ellipses. Equation (12) now has the solution

$$\left(\alpha^2 - c^2L^2\right)\frac{1}{r} = \pm c\sqrt{(LE)^2 + m^2c^2\left(\alpha^2 - L^2c^2\right)}\cosh\left(\theta\sqrt{\frac{\alpha^2}{c^2L^2} - 1}\right) - E\alpha. \quad (17)$$

In this expression, we take the positive root for $\alpha > 0$ and the negative root for $\alpha < 0$. It is obvious that, as θ increases, $(1/r)$ keeps increasing in the case of attractive motion so that the test particle spirals to the origin. This does not happen in the Kepler problem in Newtonian physics. As is well known, the angular momentum term gives a repulsive L^2/r^2 contribution to the effective potential in any central force problem. In the case of $-(1/r)$ potential, the angular momentum term prevents any particle with non-zero L from reaching the origin. This is not the case in special relativistic motion under attractive Coulomb field. If the angular momentum is less than a critical value, α/c, then the particle spirals down to the origin. You will find it useful to re-analyse this situation in terms of suitable effective potentials.

If we think of α as GMm, the second term in (12) gives a correction to the potential $(-G^2M^2/2c^2)(m/r^2)$. This term will lead to a precession of the ellipse but the model is totally wrong. One cannot represent gravity using a vector potential; in such a theory, like charges repel while the gravity has to be attractive. The proper way of generalizing the gravitational Kepler problem, taking into account the effects of relativity, is of course to use general relativity to describe the gravitational field [3]. In the case of a test particle moving in the gravitational field of a point mass at the origin, we have to take into account the effect of the curvature of the space which is described by the modification to the line interval in the form

$$\mathrm{d}s^2 = -f(r)c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\theta^2\right), \quad (18)$$

where $f(r) = [1 - (2GM/c^2r)]$. We had already discussed this in a previous installment [3] where it was pointed out that one can obtain the correct equation by replacing coordinate intervals by the proper geometrical quantities. For a relativistic *free* particle, the relationship between energy and momentum is given by $E^2 = p^2c^2 + m^2c^4$. When expressed in polar coordinates, this takes the form

$$E^2 = m^2c^4 + \left(\frac{E}{c}\right)^2 \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{c^2L^2}{r^2}. \tag{19}$$

We now have to replace in (19), $\mathrm{d}r$ by the proper length $(\mathrm{d}r/\sqrt{f})$, $\mathrm{d}t$ by the proper time $\sqrt{f}\ \mathrm{d}t$ and the energy by the redshifted one $E/\sqrt{f}$. The expression for conserved angular momentum will also change from $L = mr^2(\mathrm{d}\theta/\mathrm{d}t)$ to $L = mr^2(\mathrm{d}\theta/\sqrt{f}\mathrm{d}t)$. Manipulating these equations, it is easy to obtain an expression for $\mathrm{d}r/\mathrm{d}\theta$, differentiating which we will get the equation for the orbit in the standard form:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{GMm^2}{L^2} + \frac{3GM}{c^2}u^2. \tag{20}$$

The first term on the right-hand side is purely Newtonian and the second term is the correction from general relativity. The ratio of these two terms is $(L/mrc)^2 \approx (v/c)^2$, where r and v are the typical radius and speed of the particle.

This correction term changes the nature of the orbits in two ways. First, it changes the relationship between the parameters of the orbit and the energy and angular momentum of the particle. More importantly, it makes the elliptical orbit of Newtonian gravity to precess slowly which is of greater observational importance.

The exact solution to (20) can be given only in terms of elliptic functions and hence is not very useful. An approximate solution to (20), however, can be obtained fairly easily when the orbit has a very low eccentricity and is nearly circular (which is the case for most planetary orbits). Then the lowest order solution will be $u = u_0 =$ constant and one can find the next order correction by perturbations theory. This can be done *without* assuming that $2GMu_0/c^2 = 2GM/c^2r_0$ is small, so that the result is valid even for orbits close to the Schwarzschild radius, as long as the orbit is nearly circular.

Let the radius of the circular orbit be r_0 for which $u = (1/r_0) \equiv k_0$. For the actual orbit $u = k_0 + u_1$, where we expect the second term to be a small correction. Changing the variables from u to u_1, where $u_1 = u - k_0$, equation (20) can be written as

$$u_1'' + u_1 + k_0 = \frac{GMm^2}{L^2} + \frac{3GM}{c^2}\left(u_1^2 + k_0^2 + 2u_1k_0\right). \tag{21}$$

We now choose k_0 to satisfy the condition

$$k_0 = \frac{GMm^2}{L^2} + k_0^2\frac{3GM}{c^2}, \tag{22}$$

which determines the radius $r_0 = 1/k_0$ of the original circular orbit in terms of the other parameters. Now the equation for u_1 becomes

$$u_1'' + \left(1 - \frac{6k_0GM}{c^2}\right)u_1 = \frac{3GM}{c^2}u_1^2. \tag{23}$$

This equation is exact. We shall now use the fact that the deviation from circular orbit, characterized by u_1 is small and ignore the right-hand side of (23). Solving (23), with the right-hand side set to zero, we get

$$u_1 \cong A\cos\left[\left(1 - \frac{6GM}{c^2r_0}\right)^{1/2}\theta\right]. \tag{24}$$

We see that r does not return to its original value at $\theta = 0$ when $\theta = 2\pi$ indicating a precession of the orbit. We encountered the same phenomenon in the case of motion in a Coulomb field as well. As described in that context, the argument of the cosine function becomes 2π when

$$\theta_c \approx 2\pi[1 - (6GM/c^2r_0)]^{-1/2} \tag{25}$$

which gives the precession $(\theta_c - 2\pi)$ per orbit.

We can make a naive comparison between this precession rate and the corresponding one in the Coulomb problem by noticing that, in the latter case, we can substitute $\alpha = GMm$ and $L^2 \approx GMm^2r_0$ (which follows from (22) at the lowest order) to obtain

$$\omega_{\mathrm{el}}^2 \to 1 - \frac{GM}{c^2 r_0} \tag{26}$$

which differs by a factor 6 in the corresponding term in general relativity.

If we attempt to reproduce the general relativistic results by an effective Newtonian potential, we need to find a V_{eff} which satisfies the equation

$$-\frac{m}{L^2}\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}u} = \frac{GMm^2}{L^2} + \frac{3GM}{c^2}\,u^2 \tag{27}$$

which integrates to give

$$V_{\mathrm{eff}} = -\frac{GMm}{r} - \frac{GML^2}{mc^2}\frac{1}{r^3}\,. \tag{28}$$

The trouble with this effective potential is that the correction term depends on the angular momentum L of the particle which is somewhat difficult to motivate physically. But if you are willing to live with it, then one can introduce a pseudo Newtonian description of the general relativistic Kepler problem by taking the equations of motion to be $m(\mathrm{d}^2\mathbf{r}/\mathrm{d}\tau^2) = \mathbf{F}$ with

$$\mathbf{F} = -\hat{\mathbf{r}}\frac{GMm}{r^2}\left(1 + \frac{3(\hat{\mathbf{r}} \times \mathbf{u})^2}{c^2}\right); \qquad \mathbf{u} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau}\,, \tag{29}$$

where τ is the proper time and $\hat{\mathbf{r}}$ is a unit vector in the radial direction. You can convince yourself that the conserved angular momentum now is $\mathbf{L} = m\mathbf{r} \times \mathbf{u}$ which will ensure that the above force reproduces the correct relativistic orbit equation. Unfortunately, this force law does not seem to lead to any other useful concept.

Suggested Reading

[1] T Padmanabhan, Planets Move in Circles!, *Resonance*, Vol.1, No.9, pp.34–40, 1996.

[2] T Padmanabhan, Perturbing Coulomb to Avoid Accidents!, *Resonance*, Vol.14, No.6, pp.622–629, 2009.

[3] T Padmanabhan, Schwarzschild Metric at a discounted price, *Resonance*, Vol.13, No.4, pp.312–318, 2008.

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