Classroom

In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Geometric Insight into
Scalar Combination of Linear Equations

This article discusses the geometric significance of scalar combinations of linear equations, as a radial fan of lines passing through a point, an interpretation which, in the author’s study, has not been put forth in standard treatments of the topic. The relationship of this interpretation with solving simultaneous equations by ‘eliminating’ variables between them is also discussed.

Introduction

The notion of scalar combination of linear equations or scalar combination of rows of a matrix is quite fundamental to the study of linear algebra. The most basic lesson in linear algebra is the simultaneous solution of linear equations. However, there is an aspect of this notion that is not commonly known, the geometric aspect. This article presents a geometric interpretation of the concept of scalar combination of linear equations which, in the author’s study, has not been put forth in standard treatments of the topic.

Keywords
Linear algebra, linear dependence, linear combination, family of lines, family of planes.
Consider two equations $L_1$ and $L_2$ linear in $x$ and $y$:

\[
L_1 : a_1x + b_1y = c_1 \\
L_2 : a_2x + b_2y = c_2.
\]

$L_1$ and $L_2$ intersect at a point $P(x, y)$:

\[
x = (b_2c_1 - b_1c_2)/(a_1b_2 - a_2b_1), \quad y = (a_1c_2 - a_2c_1)/(a_1b_2 - a_2b_1)
\]

provided $a_1b_2 - a_2b_1 \neq 0$.

A scalar combination $(\alpha L_1 + \beta L_2)$ of these equations produces a third equation $L_3$ with coefficients $a_3$, $b_3$, $c_3$ different from those of $L_1$ and $L_2$:

\[
(\alpha a_1 + \beta a_2)x + (\alpha b_1 + \beta b_2), y = \alpha c_1 + \beta c_2
\]

or

\[
L_3 : a_3x + b_3y = c_3,
\]

where

\[
a_3 = \alpha a_1 + \beta a_2, b_3 = \alpha b_1 + \beta b_2, c_3 = \alpha c_1 + \beta c_2.
\]

If we plot (5), how will it look? Where will the line $L_3$ lie with respect to $L_1$ and $L_2$? What will be the slope of $L_3$? Is there a pattern in its appearance for different values of $\alpha$, $\beta$?

The short answer is that $L_3$ would resemble, as we go over all possible combinations of $\alpha$, $\beta$, a full ‘spoked bicycle wheel’ or Sun’s rays as it were, centred at the intersection of $L_1$ and $L_2$ as shown in Figure 1. In other words, $L_3$ represents the entire family of lines passing radially through the intersection of $L_1$ and $L_2$. This is guaranteed by the fact that the point of intersection (given by equation 3) satisfies (5) for any values of $\alpha$ and $\beta$, thus reducing it to an identity.

![Figure 1. The scalar combination of two lines, as the set of all lines passing through the point of intersection of the two lines.](image-url)
Conversely, every line through the point of intersection of $L_1$ and $L_2$ can be shown to be expressible as a scalar combination of $L_1$ and $L_2$. Any such line $L_3$ has the equation

$$L_3 : y - y_P = m(x - x_P),$$

(7)

where $m$ is the slope of $L_3$. We show here that even if $m$ is arbitrarily specified, there always exist $\alpha$ and $\beta$ such that $L_3$ can be written as $\alpha L_1 + \beta L_2$ for any $m$. Substituting for $x_P, y_P$ in (7) the values indicated in (3), we are able to write

$$y - \frac{(a_1 c_2 - a_2 c_1)}{(a_1 b_2 - a_2 b_1)} = m \left[ x - \frac{(b_2 c_1 - b_1 c_2)}{(a_1 b_2 - a_2 b_1)} \right]$$

or,

$$mb_2(a_1 x + b_1 y - c_1) - mb_1(a_2 x + b_2 y - c_2) - (a_1 b_2 - a_2 b_1)y + (a_1 c_2 - a_2 c_1) = 0.$$ 

By adding and subtracting $a_1 a_2 x$, it can be written as the following sum of equations:

$$(mb_2 L_1 - mb_1 L_2 + a_2 L_1 - a_1 L_2),$$

or as:

$$(mb_2 + a_2) L_1 - (mb_1 + a_1) L_2,$$

i.e., a scalar combination $\alpha L_1 + \beta L_2$ of the lines, with $\alpha = mb_2 + a_2$ and $\beta = - (mb_1 + a_1)$.

**Eliminating Variables as a Scalar Combination Operation**

The operation of ‘eliminating variables’ between equations actually consists of picking that linear combination of the equations in which the coefficient for one of the variables is zero. Since the resulting equation contains only one variable, it can be solved directly. Geometrically, we have seen above that varying the values $\alpha$ and $\beta$ from $-\infty$ to $\infty$ generates all the lines fanning out of the intersection of $L_1$ and $L_2$ at all angles. Clearly, one such pair $(\alpha, \beta)_V$ would generate the vertical line passing through $P$, and another pair $(\alpha, \beta)_H$ the horizontal line. Therefore, the linear combinations leading to the ‘elimination’ of $x$ and $y$, and yielding solutions for $y$ and $x$, respectively, are the very combinations of $\alpha$ and $\beta$ that correspond to the horizontal and vertical lines through the point of intersection. These combinations are as follows:

$$(\alpha, \beta)_H = \left(\frac{1}{a_1}, \frac{-1}{a_2}\right), \ (\alpha \beta)_V = \left(\frac{1}{b_1}, \frac{-1}{b_2}\right).$$
Linear ‘Dependence’

The ‘dependence’ of L₃ upon L₁ and L₂ means that the (simultaneous) solution of L₃ with L₁ or of L₃ with L₂ ‘depends’ upon (in being identical to) the (simultaneous) solution of L₁ and L₂. Geometrically, this dependence manifests as follows: Every line L₃ (α,β) passes through the intersection of L₁ and L₂. Conversely, any line passing through the point of intersection of two other lines can be said to be ‘dependent’ upon the two.

Slope as Function of Scalar Multipliers

Given the scalar combination \( L₃ = (αL₁ + βL₂) \) what would be the slope of L₃? What is the functional relationship between the values of the scalar multiples and the slope?

Line \( L₃ : a₃x + b₃y = c₃ \) has slope \( m = -\frac{a₃}{b₃} \). Substituting the values of \( a₃ \) and \( b₃ \) from (6), we can write: \( m = -\frac{(αa₁ + βa₂)}{(αb₁ + βb₂)} \).

It can be seen that \( m \) will depend only upon the ratio \( α/β \). Representing this ratio by \( r \), we are able to write \( m = -\frac{(ra₁ + a₂)}{(rb₁ + b₂)} \). This equation of \( m \) in terms of \( r \) represents a hyperbola centred at \(( -b₂/b₁, -a₁/b₁) \) and symmetric about a 45° line through the centre. Plotted, this looks as shown in Figure 2 for the example lines \( L₁: x + y = 10 \) and \( L₂: y - x = 1 \) (i.e., \( a₁ = 1, a₂ = -1, b₁ = 1, b₂ = 1 \)).

If \( L₁ \) and \( L₂ \) are represented in their slope–intercept forms \( y = mx + c \), the slope \( m₃ \) of \( L₃ \) comes out as \( m₃ = (αm₁ + βm₂)/(α + β) \), which is evidently the weighted average of \( m₁ \) and \( m₂ \) and clearly depicts the spoked wheel.

Figure 2. The slope ‘\( m \)’ of the scalar combination of two lines, as a function of the ratio ‘\( r \)’ of the two scalar multiples, for the example of lines \( L₁ \) and \( L₂ \) shown.
For the case where \( a_1b_2 - a_2b_1 = 0 \), \( L_1 \) and \( L_2 \) represent parallel lines, and their linear combination would represent a third line also parallel to each. In this case, it is the perpendicular distances \( \perp_1, \perp_2 \), and \( \perp_3 \) from the origin (instead of the angles) of \( L_1 \), \( L_2 \), and \( L_3 \) that would be interrelated by a hyperbolic relationship:

\[
\perp_3 = \frac{(ab_1 \perp_1 + b_2 \perp_2)}{(ab_1 + b_2)} = \frac{(r b_1 \perp_1 + b_2 \perp_2)}{(r b_1 + b_2)}.
\]

When written in the slope–intercept form (i.e., with the coefficients \( b \) normalised to 1), the relationship between the perpendicular distances takes on a form easier to understand, that of weighted averages:

\[
\perp_3 = \frac{(\alpha \perp_1 + \beta \perp_2)}{(\alpha + \beta)}.
\]

**Linear Combination of Two Planes**

The extension of the above insight to three dimensions is to consider two equations \( P_1 \) and \( P_2 \) in three variables \( x, y, z \), representing two planes.

\[a_1x + b_1y + c_1z = d_1 : P_1, \quad a_2x + b_2y + c_2z = d_2 : P_2.\]

If the left-hand sides of \( P_1 \) and \( P_2 \) are not scalar multiples of each other, these are two planes intersecting each other along a line \( L \). Indeed, the equations of \( P_1 \) and \( P_2 \) would as a pair constitute the equations of this line. A linear combination of the two equations produces a third equation, \( P_3 \), which would naturally also represent a plane. How do we expect this plane \( P_3 \) to be configured in relation to \( P_1 \) and \( P_2 \)?

![Figure 3. The scalar combination of two planes \( P_1 \) and \( P_2 \) as the set of planes \( P_3 \) passing through \( L \), the line of intersection of \( P_1 \) and \( P_2 \).](image)
In analogy with the case of the lines seen earlier, where their scalar combination represented the family of lines passing radially through their point of intersection, in this case the scalar combination would represent the family of all planes passing radially through their line of intersection, as shown in Figure 3, by rotating $P_1$ or $P_2$ about $L$. The reader is encouraged to verify this geometric intuition algebraically by using $\alpha$ and $\beta$ as the multipliers for $P_1$ and $P_2$ and proving that $P_3$ passes through $L$ for any combination of $\alpha$ and $\beta$. $P_3$ thus represents the planes generated by rotating either $P_1$ or $P_2$ about $L_1$ through the complete range of 0 to 360 degrees.

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**Illustrating the Reactivity-Selectivity Principle and the iso-Selectivity Rule through Ring Substituted alpha-azidobenzyl Carbocations**

It is axiomatic in organic chemistry that a compound or intermediate which is *highly unstable* in a thermodynamic sense is *highly reactive* in a kinetic sense and the one which is *highly stable* in a thermodynamic sense is *highly unreactive* in a kinetic sense. In this context reactivity represents the capacity of a species to react and selectivity is a mere number that represents the ratio of reaction rates.

‘Reactivity’ of a species describes how fast it reacts compared to the other species. Comparing two reactants $A_1$ and $A_2$, it is apparent that $A_1$ is more reactive than $A_2$ towards $B$, if $k_1 > k_2$ for the reactions:

$$A_1 + B \xrightarrow{k_1} \text{products}$$  \hspace{1cm} (1)

and

$$A_2 + B \xrightarrow{k_2} \text{products},$$  \hspace{1cm} (2)

where $k_1$ and $k_2$ are rate constants.

**Keywords**
Benzyl azide, reactivity-selectivity, $\alpha$-azidocation.