Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Geometric Insight into Scalar Combination of Linear Equations

This article discusses the geometric significance of scalar combinations of linear equations, as a radial fan of lines passing through a point, an interpretation which, in the author's study, has not been put forth in standard treatments of the topic. The relationship of this interpretation with solving simultaneous equations by 'eliminating' variables between them is also discussed.

Introduction

The notion of scalar combination of linear equations or scalar combination of rows of a matrix is quite fundamental to the study of linear algebra. The most basic lesson in linear algebra is the simultaneous solution of linear equations. However, there is an aspect of this notion that is not commonly known, the geometric aspect. This article presents a geometric interpretation of the concept of scalar combination of linear equations which, in the author's study, has not been put forth in standard treatments of the topic.

Keywords
Linear algebra, linear dependence, linear combination, family of lines, family of planes.

Consider two equations L_1 and L_2 linear in x and y:

$$L_1 : a_1x + b_1y = c_1 \tag{1}$$

$$L_2 : a_2x + b_2y = c_2 \ . \tag{2}$$

L_1 and L_2 intersect at a point $P(x, y)$:

$$x = (b_2c_1 - b_1c_2)/(a_1b_2 - a_2b_1)\,, \;\; y = (a_1c_2 - a_2c_1)/(a_1b_2 - a_2b_1) \tag{3}$$

provided $a_1b_2 - a_2b_1 \neq 0$.

A scalar combination $(\alpha L_1 + \beta L_2)$ of these equations produces a third equation L_3 with coefficients a_3, b_3, c_3 different from those of L_1 and L_2:

$$(\alpha a_1 + \beta a_2)x + (\alpha b_1 + \beta b_2), y = \alpha c_1 + \beta c_2 \tag{4}$$

or

$$L_3 : a_3x + b_3y = c_3\,, \tag{5}$$

where

$$a_3 = \alpha a_1 + \beta a_2, b_3 = \alpha b_1 + \beta b_2, c_3 = \alpha c_1 + \beta c_2 \ . \tag{6}$$

If we plot (5), how will it look? Where will the line L_3 lie with respect to L_1 and L_2? What will be the slope of L_3? Is there a pattern in its appearance for different values of α, β ?

The short answer is that L_3 would resemble, as we go over all possible combinations of α, β, a full 'spoked bicycle wheel' or Sun's rays as it were, centred at the intersection of L_1 and L_2 as shown in *Figure* 1. In other words, L_3 represents the entire family of lines passing radially through the intersection of L_1 and L_2. This is guaranteed by the fact that the point of intersection (given by equation 3) satisfies (5) for *any* values of α and β, thus reducing it to an identity.

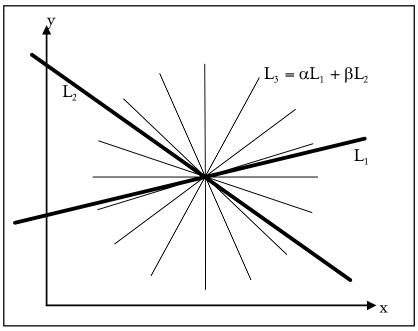

Figure 1. The scalar combination of two lines, as the set of all lines passing through the point of intersection of the two lines.

Conversely, every line through the point of intersection of L_1 and L_2 can be shown to be expressible as a scalar combination of L_1 and L_2. Any such line L_3 has the equation

$$L_3 : y - y_P = m(x - x_P) \ , \tag{7}$$

where m is the slope of L_3. We show here that even if m is arbitrarily specified, there always exist α and β such that L_3 can be written as $\alpha L_1 + \beta L_2$ for any m. Substituting for x_P, y_P in (7) the values indicated in (3), we are able to write

$$y - \frac{(a_1c_2 - a_2c_1)}{(a_1b_2 - a_2b_1)} = m\left[x - \frac{(b_2c_1 - b_1c_2)}{(a_1b_2 - a_2b_1)}\right]$$

or,

$$mb_2(a_1x + b_1y - c_1) - mb_1(a_2x + b_2y - c_2) - (a_1b_2 - a_2b_1)y + (a_1c_2 - a_2c_1) = 0\,.$$

By adding and subtracting a_1a_2x, it can be written as the following sum of equations:

$$(mb_2L_1 - mb_1L_2 + a_2L_1 - a_1L_2),$$

or as :

$$(mb_2 + a_2)L_1 - (mb_1 + a_1)L_2,$$

i.e., a scalar combination $\alpha L_1 + \beta L_2$ of the lines, with $\alpha = mb_2 + a_2$ and $\beta = -(mb_1 + a_1)$.

Eliminating Variables as a Scalar Combination Operation

The operation of 'eliminating variables' between equations actually consists of picking that linear combination of the equations in which the coefficient for one of the variables is zero. Since the resulting equation contains only one variable, it can be solved directly. Geometrically, we have seen above that varying the values α and β from $-\infty$ to ∞ generates all the lines fanning out of the intersection of L_1 and L_2 at all angles. Clearly, one such pair $(\alpha,\beta)_V$ would generate the vertical line passing through P, and another pair $(\alpha,\beta)_H$ the horizontal line. Therefore, the linear combinations leading to the 'elimination' of x and y, and yielding solutions for y and x, respectively, are the very combinations of α and β that correspond to the horizontal and vertical lines through the point of intersection. These combinations are as follows:

$$(\alpha,\beta)_H = \left(\frac{1}{a_1}, \frac{-1}{a_2}\right),\ (\alpha\beta)_V = \left(\frac{1}{b_1}, \frac{-1}{b_2}\right)\ .$$

Linear 'Dependence'

The 'dependence' of L_3 upon L_1 and L_2 means that the (simultaneous) solution of L_3 with L_1 or of L_3 with L_2 'depends' upon (in being identical to) the (simultaneous) solution of L_1 and L_2. Geometrically, this dependence manifests as follows: Every line L_3 (α,β) passes through the intersection of L_1 and L_2. Conversely, any line passing through the point of intersection of two other lines can be said to be 'dependent' upon the two.

Slope as Function of Scalar Multipliers

Given the scalar combination $L_3 = (\alpha L_1 + \beta L_2)$ what would be the slope of L_3? What is the functional relationship between the values of the scalar multiples and the slope?

Line $L_3 : a_3x + b_3y = c_3$ has slope $m = –a_3/b_3$. Substituting the values of a_3 and b_3 from (6), we can write: $m = – (\alpha a_1 + \beta a_2)/(\alpha b_1 + \beta b_2)$.

It can be seen that m will depend only upon the ratio α/β. Representing this ratio by r, we are able to write $m = – (ra_1 + a_2)/(rb_1 + b_2)$. This equation of m in terms of r represents a hyperbola centred at $(–b_2/b_1, –a_1/b_1)$ and symmetric about a 45° line through the centre. Plotted, this looks as shown in *Figure* 2 for the example lines L_1: $x + y = 10$ and L_2: $y – x = 1$ (i.e., $a_1 = 1$, $a_2 = –1$, $b_1 = 1$, $b_2 = 1$).

If L_1 and L_2 are represented in their slope–intercept forms $y = mx + c$, the slope m_3 of L_3 comes out as $m_3 = (\alpha m_1 + \beta m_2)/(\alpha + \beta)$, which is evidently the weighted average of m_1 and m_2 and clearly depicts the spoked wheel.

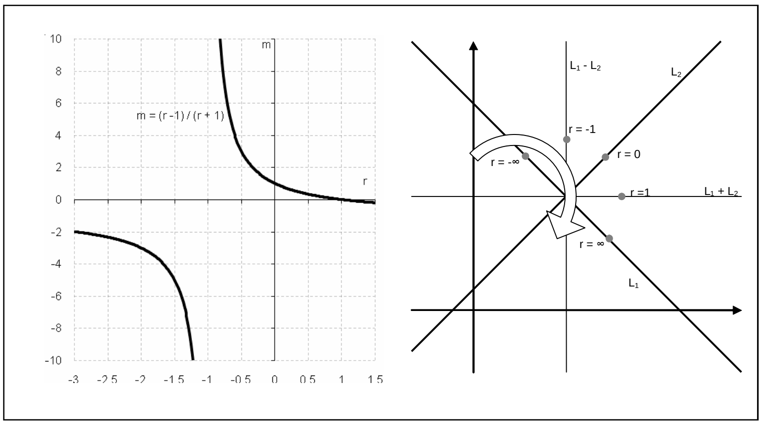

Figure 2. The slope 'm' of the scalar combination of two lines, as a function of the ratio 'r' of the two scalar multiples, for the example of lines L_1 and L_2 shown.

For the case where $a_1b_2 - a_2b_1 = 0$, L_1 and L_2 represent parallel lines, and their linear combination would represent a third line also parallel to each. In this case, it is the perpendicular distances $\perp_1$, $\perp_2$, and $\perp_3$ from the origin (instead of the angles) of L_1, L_2, and L_3 that would be interrelated by a hyperbolic relationship:

$$\perp_3 = \frac{(\alpha b_1 \perp_1 + \beta b_2 \perp_2)}{(\alpha b_1 + \beta b_2)} = \frac{(r b_1 \perp_1 + b_2 \perp_2)}{(r b_1 + b_2)}.$$

When written in the slope–intercept form (i.e., with the coefficients b normalised to 1), the relationship between the perpendicular distances takes on a form easier to understand, that of weighted averages:

$$\perp_3 = \frac{(\alpha \perp_1 + \beta \perp_2)}{(\alpha + \beta)}.$$

Linear Combination of Two Planes

The extension of the above insight to three dimensions is to consider two equations P_1 and P_2 in three variables x, y, z, representing two planes.

$$a_1x + b_1y + c_1z = d_1 : P_1, \quad a_2x + b_2y + c_2z = d_2 : P_2\,.$$

If the left-hand sides of P_1 and P_2 are not scalar multiples of each other, these are two planes intersecting each other along a line L. Indeed, the equations of P_1 and P_2 would as a pair constitute the equations of this line. A linear combination of the two equations produces a third equation, P_3, which would naturally also represent a plane. How do we expect this plane P_3 to be configured in relation to P_1 and P_2?

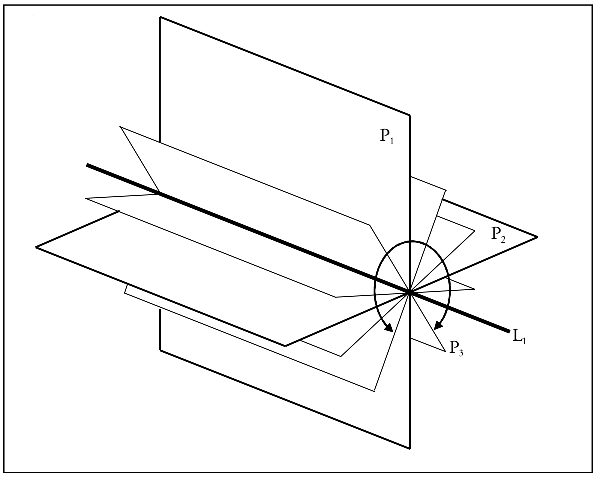

Figure 3. The scalar combination of two planes P_1 and P_2 as the set of planes P_3 passing through L, the line of intersection of P_1 and P_2

In analogy with the case of the lines seen earlier, where their scalar combination represented the family of lines passing radially through their point of intersection, in this case the scalar combination would represent the family of all planes passing radially through their *line* of intersection, as shown in *Figure* 3, by rotating P_1 or P_2 about L. The reader is encouraged to verify this geometric intuition algebraically by using α and β as the multipliers for P_1 and P_2 and proving that P_3 passes through L for any combination of α and β. P_3 thus represents the planes generated by rotating either P_1 or P_2 about L_1 through the complete range of 0 to 360 degrees.

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Illustrating the Reactivity-Selectivity Principle and the Iso-Selectivity Rule through Ring Substitutted alpha - Azidobenzyl Carbocations

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It is axiomatic in organic chemistry that a compound or intermediate which is *highly unstable* in a thermodynamic sense is *highly reactive* in a kinetic sense and the one which is *highly stable* in a thermodynamic sense is *highly unreactive* in a kinetic sense. In this context reactivity represents the capacity of a species to react and selectivity is a mere number that represents the ratio of reaction rates.

'Reactivity' of a species describes how fast it reacts compared to the other species. Comparing two reactants A_1 and A_2, it is apparent that A_1 is more reactive than A_2 towards B, if $k_1 > k_2$ for the reactions:

$$A_1 + B \xrightarrow{k_1} \text{products} \qquad (1)$$

and

$$A_2 + B \xrightarrow{k_1} \text{products}\,, \qquad (2)$$

where k_1 and k_2 are rate constants.

Keywords
Benzyl azide, reactivity-selectivity, α-azidocation.