Snippets of Physics

23. Real Effects from Imaginary Time

T Padmanabhan

Some of the curious effects in quantum theory and statistical mechanics can be interpreted by analytically continuing the time coordinate to purely imaginary values. We explore some of these issues in this instalment.

T Padmanabhan works at IUCAA, Pune and is interested in all areas of theoretical physics, especially those which have something to do with gravity.

In one of the previous instalments [1], we discussed how one can study the time evolution of a quantum wave function using a path integral propagator given by

$$K(q_2, t_2; q_1, t_1) = \sum_{\text{paths}} \exp iA[\text{path}] \ , \qquad (1)$$

where A is the classical action evaluated along a path connecting (q_1, t_1) with (q_2, t_2) and we are using units with $\hbar = 1$. This path integral kernel allows you to determine the wave function at time t_2 if it is known at time t_1 through the integral

$$\psi(q_2, t_2) = \int_{-\infty}^{+\infty} \mathrm{d}q_1 K(q_2, t_2; q_1, t_1)\psi(q_1, t_1) \ . \qquad (2)$$

These expressions are quite general. But when the Hamiltonian H describing the system is time independent, we can introduce the energy eigenfunctions through the equation $H\psi_n = E_n\psi_n$. We have also seen in the earlier instalment [1] that the kernel can be expressed in terms of energy eigenfunctions through the formula

$$K(T, q_2; 0, q_1) = \sum_n \psi_n(q_2)\psi_n^*(q_1) \exp(-iE_nT) \ . \qquad (3)$$

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So, if the energy eigenfunctions and eigenvalues are given one can determine the kernel.

directly by evaluating or approximating the path integral. The question arises as to whether one can determine the energy eigenfunctions and eigenvalues by 'inverting' the above relation. In particular, one is often interested in the ground state eigenfunction and the ground state energy of the system. Can one find this if the kernel is known?

It can be done using an interesting trick which very often turns out to be more than just a trick, having a rather perplexing domain of validity. To achieve this, let us do the unimaginable by assuming that time is actually complex and analytically continue from the real values of time t to purely imaginary values $\tau = it$. In special relativity such an analytic continuation will change the line interval from Lorentzian to Euclidean form through

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 \rightarrow \mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2 \ . \tag{4}$$

Because of this reason, one often calls quantities evaluated with analytic continuation to imaginary values of time as 'Euclidean' quantities and often denotes them with a subscript 'E' (which should not be confused with energy!). If we now do the analytic continuation of the kernel in (3) we get the result

$$K_{\mathrm{E}}(T_{\mathrm{E}}, q_2; 0, q_1) = \sum_n \psi_n(q_2)\psi_n^*(q_1)\exp(-E_n T_{\mathrm{E}}) \ . \tag{5}$$

Let us consider the form of this expression in the limit of $T_{\mathrm{E}} \rightarrow \infty$. If the energy eigenvalues are ordered as $\mathrm{E}_0 < \mathrm{E}_1 < ....$ then, in this limit, only the term with the ground state energy will make the dominant contribution and remembering that ground state wave function is real for the systems we are interested in, we get,

$$K_{\mathrm{E}}(T_{\mathrm{E}}, q_2; 0, q_1) \approx \psi_0(q_2)\psi_0(q_1)\exp(-E_0 T_{\mathrm{E}}); \ (T_{\mathrm{E}} \rightarrow \infty) \, . \tag{6}$$

Suppose we now put $q_2 = q_1 = 0$, take the logarithm of both sides and divide by T_{E}, then in the limit of $T_{\mathrm{E}} \rightarrow \infty$, we get a formula for the ground state energy:

$$-E_0 = \lim_{T_{\mathrm{E}} \rightarrow \infty} \left[\frac{1}{T_{\mathrm{E}}} \ln K_{\mathrm{E}}(T_{\mathrm{E}}, 0; 0, 0) \right] \ . \tag{7}$$

So if we can determine the kernel by some method we will know the ground state energy of the system. Once the ground state energy is known we can plug it back into the asymptotic expansion in (6) and determine the ground state wave function.

Very often, we would have arranged matters such that the ground state energy of the system is actually zero. When $E_0 = 0$ there is a nicer way of determining the wave function from the kernel by noting that

$$\lim_{T\to\infty} K(T,0;0,q) \approx \psi_0(0)\psi_0(q) \propto \psi_0(q) \ . \tag{8}$$

So the infinite time limit of the kernel – once we have introduced the imaginary time – allows determination of both the ground state wave function as well as the ground state energy. The proportionality constant of ψ_0 can be fixed by normalising the wave function.

Of course, these ideas are useful only if we can compute the kernel without knowing the wave functions in the first place. This is possible – as we discussed in [1] – whenever the action is quadratic in the dynamical variable. In that case, the kernel in real time can be expressed in the form

$$K(t_2,q_2;t_1,q_1) = N(t_1,t_2)\exp\left[i\mathcal{A}_c(t_2,q_2;t_1,q_1)\right] \ , \tag{9}$$

where $\mathcal{A}_c$ is the action evaluated for a classical trajectory and $N(t_2,t_1)$ is a normalization factor. The same ideas will work even when we can *approximate* the kernel by the above expression. We saw in the last instalment that in the semi-classical limit the wave functions can be expressed in terms of the classical action. It follows that the kernel can be written in the above form in the same semi-classical limit. If we now analytically continue this expression to imaginary values of time, then using the result in (8) we get a simple formula for the ground state wave function in terms of the Euclidean action (that is, the action for a classical trajectory obtained after analytic continuation to imaginary value of time):

$$\begin{aligned} \psi_0(q) &\propto \exp\left[-\mathcal{A}_{\rm E}\left(T_{\rm E}=\infty,0;T_{\rm E}=0,q\right)\right] \\ &\propto \exp\left[-\mathcal{A}_E\left(\infty,0;0,q\right)\right] \ . \end{aligned} \tag{10}$$

As an application of these results, consider a simple harmonic oscillator with the Lagrangian $L = (1/2)(\dot{q}^2 - \omega^2 q^2)$. The classical action with the boundary conditions $q(0) = q_i$ and $q(T) = q_f$ is given by

$$\mathcal{A}_c = \frac{\omega}{2\sin\omega T}\left[(q_i^2+q_f^2)\cos\omega T - 2q_iq_f\right] \ . \tag{11}$$

The analytic continuation will give the Euclidean action corresponding to $i\mathcal{A}_c$ to be $-\mathcal{A}_{\rm E}$ where

$$\mathcal{A}_{\rm E} = \frac{\omega}{2\sinh\omega T}\left[(q_i^2+q_f^2)\cosh\omega T - 2q_iq_f\right] \ . \tag{12}$$

Using this in (10) we find that the ground state wave function has the form

$$\psi_0(q) \propto \exp -[(\omega/2)q^2] \tag{13}$$

which, of course, is the standard result. You can also obtain the ground state energy $(1/2)\hbar\omega$ by using (7). What is amazing, when you think about it, is that the Euclidean kernel in the limit of infinite time interval has information about the ground state of the quantum system. This is the first example in which imaginary time leads to a real result!

The analytic continuation to imaginary values of time also has close mathematical connections with the description of systems in thermal bath. To see this, consider the mean value of some observable $\mathcal{O}(q)$ of a quantum mechanical system. If the system is in an energy eigenstate described by the wave function $\psi_n(q)$, then the expectation value of $\mathcal{O}(q)$ can be obtained by integrating $\mathcal{O}(q)|\psi_n(q)|^2$ over q. If the system is in a thermal bath at temperature β^{-1}, described by a canonical ensemble, then the mean value has to be computed by averaging over all the energy eigenstates *as well* with a weightage $\exp(-\beta E_n)$. In this case, the mean value can be expressed as

$$\begin{aligned}\langle\mathcal{O}\rangle &= \frac{1}{Z}\sum_n\int \mathrm{d}q\,\psi_n(q)\mathcal{O}(q)\psi_n^*(q)\,e^{-\beta E_n} \\ &\equiv \frac{1}{Z}\int \mathrm{d}q\,\rho(q,q)\mathcal{O}(q) \ ,\end{aligned} \tag{14}$$

where Z is the partition function and we have defined a *density matrix* $\rho(q,q')$ by

$$\rho(q,q') \equiv \sum_n \psi_n(q)\psi_n^*(q')\,e^{-\beta E_n} \tag{15}$$

in terms of which we can rewrite (14) as

$$\langle\mathcal{O}\rangle = \frac{\mathrm{Tr}\,(\rho\mathcal{O})}{\mathrm{Tr}\,(\rho)} \ , \tag{16}$$

where the trace operation involves setting $q = q'$ and integrating over q. This standard result shows how $\rho(q, q')$ contains information about both thermal and quantum mechanical averaging. In fact, the expression for the density matrix in (15) is just the coordinate basis representation of the matrix corresponding to the operator $\rho = \exp(-\beta H)$. That is,

$$\rho(q, q') = \langle q|e^{-\beta H}|q'\rangle \ . \tag{17}$$

But what is interesting is that we can now relate the density matrix of a system in finite temperature – something very real and physical – to the path integral kernel in imaginary time. This is obvious from comparing (15) with (3). We find that the density matrix can be immediately obtained from the Euclidean kernel by:

$$\rho(q, q') = K_{\mathrm{E}}(\beta, q; 0, q') \ . \tag{18}$$

What is surprising now is that the imaginary time is being identified with the inverse temperature. Very crudely, this identification arises from the fact that thermodynamics in canonical ensemble uses $e^{-\beta H}$ while the standard time evolution in quantum mechanics uses e^{-itH}. But beyond that, it is difficult to understand in purely physical terms why imaginary time and real temperature should have anything to do with each other.

In obtaining the expectation values of operators which depend only on q – like the ones used in (14) – we only need to know the diagonal elements $\rho(q, q) = K_{\mathrm{E}}(\beta, q; 0, q)$. The kernel in the right hand side can be thought of as the one corresponding to a periodic motion in which a particle starts and ends at q in a time interval β. In other words, periodicity in imaginary time is now linked to finite temperature.

Believe it or not, most of the results in black hole thermodynamics can be obtained from this single fact by noting that the spacetimes representing a black hole, for example, have the appropriate periodicity in imaginary time. Considering the elegance of this result, let us pause for a moment and see how it comes about. Consider a curved spacetime in general relativity which has a line interval

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + \mathrm{d}L_\perp^2 \ , \tag{19}$$

where $\mathrm{d}L_\perp^2$ represents metric in two transverse directions. For example, we saw in a previous instalment [2] that the Schwarzschild metric representing a black hole has this form with $f(r) = 1 - (r_g/r)$, where $r_g = (2GM/c^2) = 2M$ (in units with

$G = c = 1$) and $\mathrm{d}L_\perp^2$ represents the standard metric on a two sphere. The only property we will actually need is that $f(r)$ has a simple zero at some $r = a$ with $f'(a) \equiv 2\kappa$ being some constant. In the case of the black hole metric, $\kappa = (1/2r_g)$. When we consider the metric near the horizon $r \approx a$, we can expand $f(r)$ in a Taylor series and reduce it to the form

$$\mathrm{d}s^2 = -2\kappa l \mathrm{d}t^2 + \frac{\mathrm{d}l^2}{2\kappa l} + \mathrm{d}L_\perp^2 \ , \tag{20}$$

where $l \equiv (r - a)$ is the distance from the horizon. If we now make a coordinate transformation from l to another spatial coordinate x such that $(\kappa x)^2 = 2\kappa l$, the metric becomes

$$\mathrm{d}s^2 = -\kappa^2 x^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}L_\perp^2 \ . \tag{21}$$

This represents the metric near the horizon of a black hole.

So far we have not done anything non-trivial. Now we shall analytically continue to imaginary values of time with $it = \tau$ and denote $\kappa\tau = \theta$. Then the corresponding analytically continued metric becomes

$$\mathrm{d}s^2 = x^2 \mathrm{d}\theta^2 + \mathrm{d}x^2 + \mathrm{d}L_\perp^2 \ . \tag{22}$$

But $\mathrm{d}x^2 + x^2\mathrm{d}\theta^2$ is just the metric on a two-dimensional plane in polar coordinates and if it has to be well behaved at $x = 0$, the coordinate θ must be periodic with period 2π. Since $\theta = \kappa\tau$, it follows that the imaginary time τ must be periodic with period $2\pi/\kappa$ as far as any physical phenomenon is concerned. But we saw earlier that such a periodicity of the imaginary time is mathematically identical to working in finite temperature with the temperature

$$\beta^{-1} = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_g} = \frac{\hbar c^3}{8\pi G M} \ , \tag{23}$$

where the first equality is valid for a general class of metrics (with suitably defined κ by Taylor expansion) while the last two results are for the Schwarzschild metric and in the final expression we have reverted back to normal units. This is precisely the Hawking temperature of a black hole of mass M which we obtained by a different method in a previous instalment [3]. Here we could do that just by looking at the form of the metric near the horizon and using the relation between periodicity in imaginary time and temperature. While these results have been verified by several other methods in the context of general relativity, a transparent physical understanding is still lacking.

The imaginary time and Euclidean action also play an interesting role in the case of tunneling. To see this, let us start with the expression for the classical action written in a slightly different form:

$$A = \int \mathrm{d}tL = \int \mathrm{d}t(p\dot{q} - H) = \int p\mathrm{d}q - \int H\mathrm{d}t \ . \tag{24}$$

While using the action principle, we usually concentrate on trajectories with fixed end points (t_1, q_1) and (t_2, q_2). When the Hamiltonian is independent of time, we can also study classical trajectories of particles with a fixed value for energy E. In this case, the second term in (24) becomes just Et and the non-trivial variation actually comes from the first term. Expressing p as $\sqrt{2m(E-V)}$ for a particle of mass m moving in a potential V, we get an action – closely related to what is called 'Jacobi action' – given by

$$S = \int p\mathrm{d}q = \int \sqrt{2m(E-V)}\mathrm{d}q \ . \tag{25}$$

As long as $E > V$, this will lead to a real value for S. Tunneling occurs, however, when $E < V$. To simplify matters a little bit, let us consider the case of a particle with $E = 0$ (which can always be achieved by a constant to the Hamiltonian) moving in a potential $V > 0$. In that case the action becomes pure imaginary and is given by

$$S = i\int \sqrt{2mV}\,\mathrm{d}q \ , \tag{26}$$

and the corresponding branch of the semi classical wave function (studied in the last instalment) will be exponentially damped:

$$\psi \propto \exp(\mathrm{iS}) = \exp - \left(\int \sqrt{2mV}\,\mathrm{d}q\right) \ . \tag{27}$$

This represents the fact that you cannot have a classical trajectory with $E = 0$ in a region in which $V > 0$.

It is however possible to have such a trajectory if we analytically continue to imaginary values of time. In real time, the conservation of energy for a particle with $E = 0$ gives $(1/2)m(\mathrm{d}q/\mathrm{d}t)^2 = -V(q)$ which cannot have real solutions when

$V > 0$. But when we set $t = -i\tau$ this equation becomes $(1/2)m(\mathrm{d}q/\mathrm{d}\tau)^2 = V(q)$ which, of course, has perfectly valid solutions when $V > 0$. So the tunneling through a potential barrier can be interpreted as a particle moving off to imaginary values of time as far as the mathematics goes. The Euclidean action will now be

$$S_{\mathrm{E}} = \int \sqrt{2mV}\ \mathrm{d}q\ . \tag{28}$$

All that we need to do to obtain the tunneling amplitude is to replace iS by $-S_{\mathrm{E}}$ in the argument of the relevant exponential so that the wave function in (27) becomes:

$$\psi \propto \exp iS = \exp - \left(\int \sqrt{2mV}\, dq \right) = \exp(-S_{\mathrm{E}})\ . \tag{29}$$

So we find that the tunnelling amplitude across the potential can also be related to analytic continuation in the imaginary time and and the Euclidean action.

Finally we will use these ideas to obtain a really non-trivial phenomenon in quantum electrodynamics, called the Schwinger effect, named after Julian Schwinger who was one of the creators of quantum electrodynamics and received a Nobel Prize for the same. In simplest terms, this effect can be stated as follows. Consider a region of space in which there exists a constant, uniform electric field. One way to do this is to set-up two large, parallel, conducting plates separated by some distance L and connect them to the opposite poles of a battery. This charges the plates and produces a constant electric field between them. Schwinger showed that, in such a configuration, electrons and positrons will spontaneously appear in the region between the plates through a process which is called pair production from the vacuum.

The first question one would ask is how particles can appear out of nowhere. This is natural since we haven't seen tennis balls or chairs appear out of the vacuum spontaneously. In quantum field theory, what we call 'vacuum' is actually bristling with quantum fluctuations of the fields which can be interpreted in terms of virtual particle-antiparticle pairs (see [4]). Under normal circumstance, such a virtual electron-positron pair will be described by the situation in the left frame of *Figure* 1. We think of an electron and positron being created at the event A and then getting annihilated at the event B. In the absence of any external fields, there is no force acting on these virtual pairs and they continuously appear and disappear quite randomly in the spacetime.

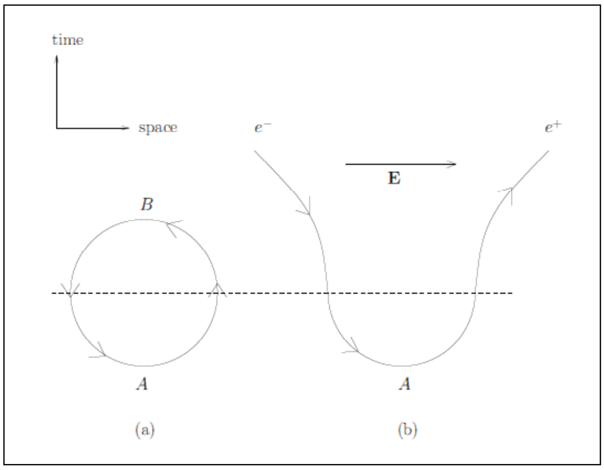

Figure 1. In the vacuum there will exist virtual electron-positron pairs which are constantly created and annihilated as shown in the left frame (a) An electron-positron pair is created at A and annihilated at B with the positron being interpreted as an electron going backward in time. The right frame (b) shows how, in the presence of an electric field, this virtual process can lead to creation of real electrons and positrons.

Consider now what happens if there is a electric field present in this region of space. The electric field will pull the electron in one direction and push the positron in the opposite direction since the electrons and positrons carry opposite charges. In the process the electric field will do work on the virtual particle-antiparticle pair and hence will supply energy to them. If the field is strong enough, it can supply an energy greater than the rest energy of the two charged particles which is just $2 \times mc^2$ where m is the mass of the particle. This allows the virtual particles to become real. That is how the constant electric field between two conducting parallel plates produces particles out of the vacuum. It essentially does work on the virtual electron-positron pairs which are present in the spacetime and converts them into real particles as shown in the right frame of *Figure* 1(b). One way to model this is to assume that the the particle tunnels from the trajectory on the left to the one on the right through the semicircular path in the lower half. The trajectories on the left and right are real trajectories for the charged particle but the semicircle is a 'forbidden' quantum process. We will now see how imaginary time makes this possible.

To do this, we begin with the trajectory in real time which will correspond to relativistic motion with uniform acceleration $g = qE/m$. We have worked this out in a previous instalment [3] and the result is given – with suitable choice of initial conditions – by

$$x = (1/g)\cosh(g\tau); \;\; t = (1/g)\sinh(g\tau); \;\; x^2 - t^2 = 1/g^2 \; . \qquad (30)$$

The trajectory is a (pair of) hyperbola in the $t - x$ plane shown in *Figure* 1(b). If we now analytically continue to imaginary values of τ and t, the trajectory becomes a circle $x^2 + t_{\mathrm{E}}^2 = 1/g^2$ of radius $(1/g)$ and the parametric equations become

$$x = (1/g)\cos\theta; \qquad t = (1/g)\sin\theta; \qquad \theta = g\tau_{\mathrm{E}}\ . \tag{31}$$

By going from $\theta = \pi$ to $\theta = 2\pi$, say, we can get this to be a semicircle connecting the two hyperbolas.

To obtain the amplitude for this process we have to evaluate the value of the Euclidean action for the semicircular track. The action for a particle of charge q in a constant electric field E represented by a scalar potential $\phi = -Ex$ is given by

$$A = -m\int \mathrm{d}\tau + qE\int x\mathrm{d}t\,, \tag{32}$$

where τ is the proper time of the particle. So, on analytic continuation we get

$$\begin{aligned} iA = \; & - \; im\int \mathrm{d}\tau + iqE\int x\mathrm{d}t \\ & \rightarrow \; -m\int \mathrm{d}\tau_{\mathrm{E}} + qE\int x\mathrm{d}t_{\mathrm{E}} \equiv -A_{\mathrm{E}}\,. \end{aligned} \tag{33}$$

The Euclidean action A_E in (33) can be easily transformed to an integral over θ and noting that the integral over $x\mathrm{d}t_{\mathrm{E}}$ is essentially the area enclosed by the curve, which is a semicircle of radius $(1/g)$, we get

$$-A_{\mathrm{E}} = -(m/g)\int_{\pi}^{2\pi} \mathrm{d}\theta + (m/2g)\int_{\pi}^{2\pi} \mathrm{d}\theta = -(m\pi/2g)\ . \tag{34}$$

The limits of the integration are so chosen that the path in the imaginary time connects $x = -(1/g)$ with $x = (1/g)$ thereby allowing a virtual semi-circular loop to be formed as shown in *Figure* 1(b). Hence the final result for the Euclidean action for this classically forbidden process is

$$A_{\mathrm{E}} = \frac{\pi m}{2g} = \frac{\pi m^2}{2qE}\ . \tag{35}$$

With the usual rule that a process with $\exp \mathrm{iA}$ gets replaced by $\exp(-A_{\mathrm{E}})$ when it is classically forbidden, we find the amplitude for this process to take place to be $\mathcal{A} \propto \exp(-A_{\mathrm{E}})$. The corresponding probability $\mathcal{P} = |\mathcal{A}|^2$ is given by

$$\mathcal{P} \approx \exp -(\pi m^2/qE)\ . \tag{36}$$

This is the leading term for the probability which Schwinger obtained for the pair creation process. (In fact, one can even obtain the sub-leading terms by transfering paths which wind around several times in the circle but we will not go into it; if you are interested, take a look at ref.[5]). Once again the moral is clear. What is forbidden in real time is allowed in imaginary time!

Suggested Reading

[1] T Padmanabhan, The Optics of Particles, *Resonance*, Vol.14, pp.8–18, 2009.

[2] T Padmanabhan, Schwarzschild Metric at a discounted price, *Resonance*, Vol.13, pp.312–318, 2008.

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[5] K Srinivasan and T Padmanabhan, Particle Production and Complex Path Analysis, *Phys. Rev.* D, Vol.60, p.24007, 1999.

Address for Correspondence: T Padmanabhan, IUCAA, Post Bag 4, Pune, University Campus, Ganeshkhind, Pune 411 007, India. Email: paddy@iucaa.ernet.in, nabhan@iucaa.ernet.in