Snippets of Physics

21. Extreme Physics

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Variational principles play a central role in theoretical physics in many guises. We will discuss, in this instalment, some curious features associated with a couple of variational problems.

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In Herman Melville's 1851 classic *Moby Dick* there is a chapter called "The Try-Works" which describes how the try-pots of the ship *Pequod* are cleaned. (In case you haven't read the book, a try-pot is a large cauldron, usually made of iron, which is used to obtain liquid oil from whale blubber.) In that is the passage: "It was in the left hand try-pot of the Pequod ..... that I was first indirectly struck by the remarkable fact, that in geometry all bodies gliding along the cycloid, my soapstone for example, will descend from any point in precisely the same time."

The remarkable fact Melville writes about is related to what is known as the brachistochrone problem (*brachistos* meaning shortest and *chronos* referring to time) which requires us to find a curve connecting two points A and B in a vertical plane such that a particle, sliding under the action of gravity, will travel from A to B in the shortest possible time. It was known to Johann Bernoulli (and to several others; see *Box* 1 for a taste of history) that this curve is (a part of) a cycloid if we take the Earth's gravitational field to be constant. The cycloidal path also has the property that the time taken for a particle to roll from any point to the minimum of the curve is independent of where it started from – which is what Melville was talking about. In other words, a particle executing oscillations in a cycloidal track under the action of gravity will maintain a period which is

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independent of amplitude. This is quite valuable in the construction of pendulum clocks and the early clockmakers knew this well. (This earned cycloid the names isochrone and tautochrone as if brachistochrone is not enough!)

The cycloid is the curve traced by a point on the circumference of a wheel which is rolling without slipping along a straight line. From this it is easy to show (see *Figure* 1) that the parametric equation $(x = x(\theta),\ y = y(\theta))$ to a cycloid has the form

$$x = a(\theta - \sin\theta); \qquad y = a(1 - \cos\theta)\ , \tag{1}$$

where a is the radius of the rolling circle. We shall now take a closer look at this result.

While the initial solution to the brachistochrone problem involved some of the intellectual giants of the seventeenth century, it is now within the grasp of an undergraduate student. Let $y(x)$ denote the equation to the curve which is the solution to the brachistochrone problem with the coordinates chosen such that x is horizontal and y is measured vertically downwards as in *Figure* 1. Let the particle begin its slide from the origin with zero velocity. If the infinitesimal arc length along the curve around the point $\mathrm{P}(x, y)$ is $\mathrm{d}s = (1 + y'^2)^{1/2}\mathrm{d}x$, where $y' = (\mathrm{d}y/\mathrm{d}x)$, then the particle takes time $\mathrm{d}t = \mathrm{d}s/v$, where $v = \sqrt{2gy}$ is its speed at P. To determine the curve we only need to find the extremum of the integral over $\mathrm{d}t$ which is a straightforward problem in the calculus of variation. We will, however, analyse it from two slightly different approaches.

In the first approach, we shall make a coordinate transformation which simplifies the problem considerably. Let us introduce two new coordinates α and β in place of the standard Cartesian coordinates (x, y) in the first quadrant by the relations

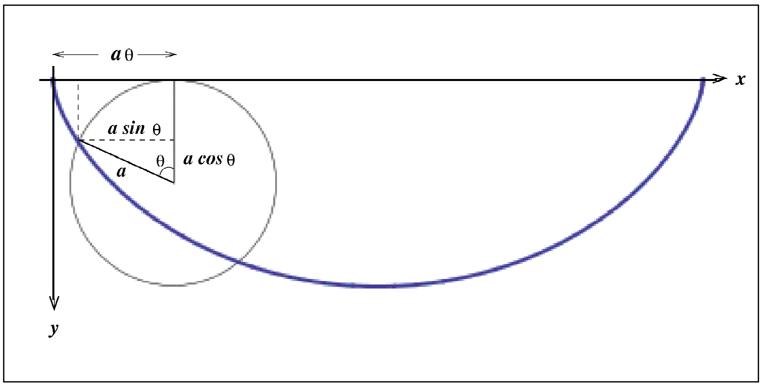


Figure 1.

$$x = \alpha^2 \left(\frac{\beta}{\alpha} - \sin \frac{\beta}{\alpha} \right); \qquad y = \alpha^2 \left(1 - \cos \frac{\beta}{\alpha} \right) \ , \tag{2}$$

where $\alpha > 0$ and $0 \leq \beta \leq 2\pi\alpha$. Obviously, for a fixed α, the curve $x(\beta), y(\beta)$ is a cycloid (which tells you that we are cheating a little bit using our knowledge of the final solution!). The square of the velocity of the particle

$$v^2 = 2gy = \dot{x}^2 + \dot{y}^2 \ , \tag{3}$$

where overdots denote differentiation with respect to time, can now be expressed in terms of $\dot{\beta}$ and $\dot{\alpha}$ by straightforward algebra. This gives the relation

$$2gy = 2y\dot{\beta}^2 + 4 \left(2\alpha \sin \frac{\beta}{2\alpha} - \beta \cos \frac{\beta}{2\alpha} \right)^2 \dot{\alpha}^2 \ . \tag{4}$$

The term involving $\dot{\alpha}^2$ is non-negative; further, since $y > 0$ we have $\dot{\beta} \leq \sqrt{g}$. Integrating this relation between $t = 0$ and $t = T$, where T is the time of descent, we get

$$\beta(T) = \int_0^T \dot{\beta} \, \mathrm{d}t \leq \int_0^T \sqrt{g} \, \mathrm{d}t = \sqrt{g} \, T \ . \tag{5}$$

It follows that the time of descent is bounded from below by the equality $T = \beta(T)/\sqrt{g}$. The best we can do is to set $\dot{\beta} = \sqrt{g}$ and $\dot{\alpha} = 0$ to satisfy (4) and hit the lower bound in (5). Since the required curve has $\alpha =$ constant, it is obviously a cycloid parameterized by β.

The angular parameter $\theta = \beta/\alpha$ of the cycloid varies with time at a constant rate $\dot{\theta} = \dot{\beta}/\alpha = \sqrt{g}/\alpha$. It is clear from the parameterization in (2) that the radius a of the circle which rotates to generate the cycloid is related to α by $a = \alpha^2$. Hence the angular velocity of the rolling circle is $\omega = \dot{\theta} = \sqrt{g/a}$. If the particle moves all the way to the other end of the cycloid at a horizontal distance $L = 2\pi a$, then the time of flight will be $T = 2\pi/\omega = (2\pi L/g)^{1/2}$. If L is the 100 m, then with $g = 9.8\text{m s}^{-2}$ we get $T \approx 8$ sec which is better than the world record for 100 m dash! Gravity seems to do quite well.

There is another indirect way of arriving at the cycloidal solution that is of some interest. This approach uses the concept of hodograph which is the curve traced by a particle in the velocity space [1]. Let us try to determine the hodograph corresponding to the motion of swiftest descent. For simplicity, consider the full transit of the particle from a point A to a point B in the same horizontal axis $y = 0$. Let the speed of the particle be v when the velocity vector makes an angle θ with respect to the v_x-axis in the velocity space. Then the hodograph is given by some curve $u(\theta)$ which we are trying to determine. Using $\dot{x} = v\cos\theta, \dot{y} = v\sin\theta, y = v^2/2g$, we can write the relations

$$\mathrm{d}t = \frac{\mathrm{d}v}{g\sin\theta}; \qquad \mathrm{d}x = \frac{v\mathrm{d}v}{g}\cot\theta \ . \tag{6}$$

We are now required to minimize the integral over dt while keeping the integral over dx fixed. Incorporating the latter constraint by a Lagrange multiplier $(-\lambda)$, we see that we need to minimize the following integral

$$I = \int \frac{\mathrm{d}v}{g}\left(\frac{1}{\sin\theta} - \lambda v\cot\theta\right) \ . \tag{7}$$

The minimization is trivial since no derivatives of the functions are involved and leads to the relation $v = (1/\lambda)\cos\theta$ with $-\pi/2 < \theta < \pi/2$. We can now trade off the Lagrange multiplier λ for the total horizontal distance L (obtained by integrating dx) and obtain $\lambda^2 = \pi/2gL$. Hence, our hodograph is given by the equation

$$v(\theta) = \sqrt{\frac{2gL}{\pi}}\cos\theta \equiv 2R_0\cos\theta \ . \tag{8}$$

This is just the polar equation for a circle of radius R_0 with the origin coinciding with the left-most point of the circle.

How do we get to the curve in real space from the hodograph in the velocity space? In this particular case, it is quite easy. Suppose we shift the circular hodograph horizontally to the left by a distance R_0. This requires subtracting a horizontal velocity which is numerically equal to the radius of the hodograph. After the shift, we obtain the hodograph of a uniform circular motion which is, of course, a circular hodograph with the origin at its centre. Hence the motion that minimizes the time of descent is just uniform circular motion added to a rectilinear uniform motion with a velocity equal to that of circular motion. This is, of course, the path traced by a point on a circle that rolls on a horizontal surface, which is a cycloid. This approach has the advantage that you obtain the cycloid not in terms of equations but in terms of its geometrical definition.

There is an interesting generalization of the brachistochrone problem which has not received much attention. The cycloid solution was obtained under the assumption of a uniform, constant gravitational field of a flat Earth. In reality, of course, the gravitational field varies as $(1/r^2)$ around a spherical object. The question arises as to how the curve of swiftest descent gets modified when we work with the $(1/r^2)$ force.

To tackle this problem, it is convenient to use polar coordinates in the plane of motion and approximate the gravitational source as a point particle of mass M at the origin. We are interested in determining the curve $r(\theta)$ such that a particle starting from a point A (with coordinates $r = R$ and $\theta = 0$) will reach a point B (with coordinates $r = r_f,\ \theta = \theta_f$) in the shortest possible time. As usual, some new curiosities creep in.

The mathematical formulation of the variational principle is trivial. If $v(r)$ is the speed of the particle when it is at the radial distance r, then

$$v^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) = C^2\left(\frac{1}{x} - 1\right) , \tag{9}$$

where $x = r/R$ and $C^2 = 2GM/R$. The variational principle requires us to minimize the integral over $\mathrm{d}s/v$ where $\mathrm{d}s = R\mathrm{d}\theta(x'^2 + x^2)^{1/2}$ is the arc length along the curve with $x' = \mathrm{d}x/\mathrm{d}\theta$. This, in turn, requires determining the extremum of the integral

$$T = \frac{R}{C}\int \mathrm{d}\theta \left(\frac{x'^2 + x^2}{(1/x) - 1}\right)^{1/2} \equiv \int L(x', x)\mathrm{d}\theta . \tag{10}$$

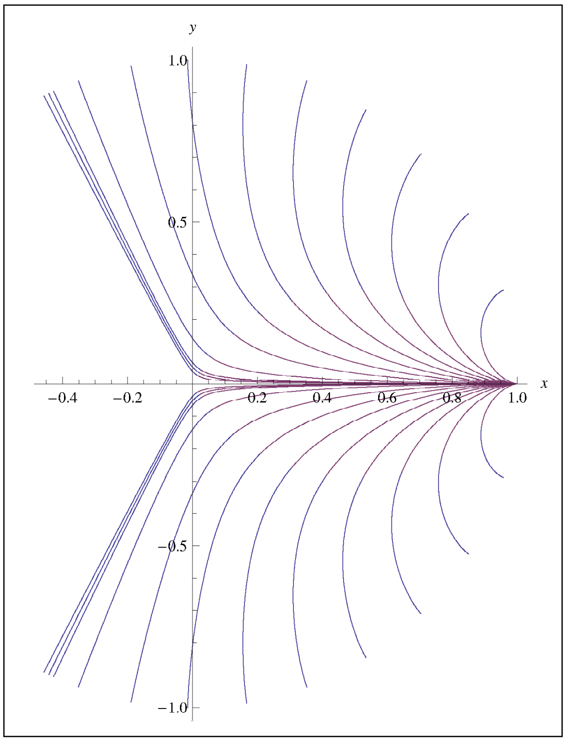
The Euler–Lagrange equation will as usual lead to a second order differential equation involving $x''(\theta)$. But since the integrand is independent of θ (time), we know that $x'(\partial L/\partial x') - L$ is conserved (energy). Equating it to a constant K gives a first integral thereby allowing the problem to be reduced to quadrature. Fairly straightforward algebra then leads to the form of the function $\theta(x)$ given by the integral

$$\theta(x) = \int_1^x \frac{\mathrm{d}\bar{x}}{\bar{x}} \sqrt{\frac{1 - \bar{x}}{\lambda \bar{x}^3 + \bar{x} - 1}} , \tag{11}$$

where $\lambda \equiv (R/KC)$.

Unfortunately, this is an elliptic integral (and a pretty bad one at that) which makes further analytic progress difficult. Working things out numerically, one

Figure 2.



can plot the relevant curves which show a very interesting behaviour (see *Figure* 2). To begin with, one notices that each curve has a turning point $x = l$, say, where $(\mathrm{d}x/\mathrm{d}\theta) = 0$. This is a point of minimum approach related to λ by $\lambda = l^{-3}(1 - l)$. What is curious is the asymptotic behaviour of the curve *after* it turns around. It is clear from *Figure* 2 that the curves never enter the 'forbidden region' between $\theta = -2\pi/3$ and $\theta = +2\pi/3$. With some hard work, one can actually prove this result analytically from the form of the integral (11). (Try it out yourself; determine the angle $\theta(l)$ at the point of minimum approach from (11) and then carefully evaluate the $l \to 0$ limit of $\theta(l)$). In fact, the 3 in $(2\pi/3)$ of the forbidden zone comes from the power law index of the force. For the brachistochrone problem in r^{-n} force law, the forbidden zone is given by $-2\pi/(n+1) < \theta < 2\pi/(n+1)$. The path of quickest descent from $r = R, \theta = 0$ to any point in the forbidden zone must necessarily pass through the singularity at the origin.

Having described the classic variational problem which started it all, I want to discuss another one which does not even seem to have a respectable name. This problem [2] can be stated as follows. Consider a planet of a given mass M and volume V and a constant density $\rho = M/V$. We are asked to vary the shape of

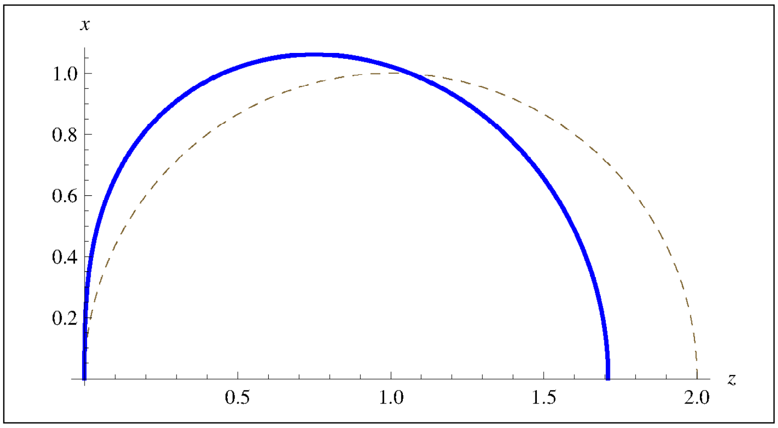


Figure 3.

the planet so as to make the gravitational force exerted by the planet on a given point at its surface the maximum possible value. What is the resulting shape?

Most people would guess that the shape is either a sphere or something like the apex of a cone. The second guess is easy to refute since it puts afair amount of the mass away from the chosen point; but a sphere remains an intriguing possibility. The correct answer, however, is quite strange and can be obtained as follows.

Let the chosen point be at the origin and let z-axis be along the direction of the maximal force acting on a test particle at the origin. It is obvious that this z-axis must be an axis of symmetry for the planet; if it is not, then one can increase the z-component of the net force by moving material from larger to smaller transverse distance until the planet is axially symmetric. Thus our problem reduces to determining the curve $x = x(z)$ (with $0 < z < z_0$, say) which, on revolution around the z-axis generates the surface of the planet. (The solution is plotted as a thick unbroken curve in *Figure* 3.)

The easiest way to calculate the z-component of the gravitational force acting on the origin is to divide the planet into circular discs each of thickness dz, located perpendicular to the z-axis. To get the force exerted on a test particle of mass m by any single disc, we further divide it into annular rings of inner radii x and outer radii $x + \mathrm{d}x$. The force along the z-axis by any one such ring will be given by

$$\mathrm{d}F = Gm(\rho 2\pi x\,\mathrm{d}x)\,\mathrm{d}z\,\frac{1}{x^2+z^2}\,\frac{z}{\sqrt{x^2+z^2}}\;. \qquad (12)$$

Hence the total force is given by

$$
\begin{aligned}
F &= 2\pi G m\rho \int_0^{z_0} \mathrm{d}z \int_0^{x(z)} x\,\mathrm{d}x \frac{z}{(x^2+z^2)^{3/2}} \\
&= \frac{3GMm}{2a^3} \int_0^{z_0} \mathrm{d}z \left(1 - \frac{z}{(x^2(z)+z^2)^{1/2}}\right) . \qquad (13)
\end{aligned}
$$

In arriving at the last expression we have expressed the density as $\rho = 3M/4\pi a^3$ so that the volume of the planet is constrained by the condition

$$
V = \pi \int_0^{z_0} \mathrm{d}z\, x^2(z) = \frac{4\pi a^3}{3} . \qquad (14)
$$

Imposing this condition by a Lagrange multiplier $(-\lambda)$, we see that we have to essentially find the extremum of the integral over the function

$$
L = 1 - \frac{z}{(x^2+z^2)^{1/2}} - \lambda x^2 . \qquad (15)
$$

This is straightforward and we get

$$
\frac{z}{(x^2+z^2)^{3/2}} = 2\lambda = \frac{1}{z_0^2} , \qquad (16)
$$

where the last equality determining the Lagrange multiplier follows from the condition that $x(z_0) = 0$. Our constraint on the total volume by (14) implies that $z_0^3 = 5a^3$ thereby completely solving the problem. The polar equation to the curve is $r^2 = 5^{2/3}a^2\cos\theta$; for comparison, a sphere with the same volume will correspond to $r = 2a\cos\theta$.

The shape of our weird planet is shown in *Figure* 3 by a thick unbroken curve (along with that corresponding to a sphere of same volume). It has no cusps at the poles and I am not aware of any specific name for this surface. The total force exerted by this planet at the origin happens to be

$$
F = \left(\frac{27}{25}\right)^{1/3} \frac{GMm}{a^2} \approx 1.03 \frac{GMm}{a^2} \qquad (17)
$$

which is not too much of a gain over a sphere. But then, as they say, it is the principle that matters.

There is a minor subtlety we glossed over while doing the variation in this problem. Unlike the usual variational problems, the end point z_0 is not given to us as fixed while doing the variation of the integrals in (13) and (14). It is possible to take

Box 1. A Bit of History

One of the early investigations about the time of descent along a curve was by Galileo. He, like many others, was interested in the time taken by a particle to perform an oscillation on a circular track which, of course, is what a simple pendulum of length L hanging from the ceiling will do. Today we could write down this period of oscillation as

$$T = \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \ , \tag{1}$$

where k is related to the angular amplitude of the swing. Of course, in the days before calculus, the expression would not have meant anything! Instead, Galileo used an ingenious geometrical argument and – in fact – thought that he has proved the circle to be the curve of fastest descent. It was, however, known to mathematicians of the 17th century that Galileo's argument did not establish such a result.

The major development came when Bernoulli threw a challenge in 1697 in the form of the brachistochrone problem to the mathematicians of that day with the interesting announcement:

"I, Johann Bernoulli, greet the most clever mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to earn the gratitude of the entire scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall the publicly declare him worthy of praise".

Bernoulli, of course, knew the answer and the problem was also solved by his brother Jakob Bernoulli, Leibniz, Newton and L' Hospital. Newton is said to have received Bernoulli's challenge at the Royal Society of London one afternoon and (according to second hand sources, like John Conduit – the husband of Newton's niece), Newton solved the problem by night-fall. The "solution", which was simply a description of how to construct the relevent cycloid, was published anonymously in the *Philosophical Transactions of the Royal Society* of January 1697 (back dated by the editor Edmund Halley). Newton actually read aloud his solution in a Royal Society meeting only on 24 February 1697. Legend has it that Bernoulli had immediately recognized Newton's style and exclaimed "*tanquam ex ungue leonem*" meaning 'the lion is known by its claw'!

this into account by a slightly more sophisticated treatment but it will lead to the same result in this particular case.

Suggested Reading

[1] T Padmanabhan, See e.g., Planets move in circles! A different view of orbits, *Resonance*, Vol. 1, No.9, p.34, 1996 for a description of hodograph in the case of Kepler problem.

[2] W D Macmillan, *Theory of the Potential*, Dover Publications, 1958.

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