

## Snippets of Physics

### 17. Why does an Accelerated Charge Radiate?

*T Padmanabhan*

The fact that an accelerated charge radiates energy is considered an elementary textbook result in electromagnetism. Nevertheless, this process of radiation (and its reaction on the charged particle) raises several conundrums about which technical papers are written even today. In this installment, we will try to understand why an accelerated charge radiates in a simple, yet rigorous, manner.

The electric field of a point charge at rest at the origin falls as  $(1/r^2)$  and is directed radially outward from the charge. If the charge moves with a uniform velocity  $\mathbf{v}$ , the field is given by

$$\mathbf{E} = \frac{q\mathbf{r}}{r^3} \frac{(1 - v^2/c^2)}{(1 - (v^2/c^2) \sin^2 \theta)^{3/2}}; \quad \mathbf{B} = \frac{1}{c} \mathbf{v} \times \mathbf{E}, \quad (1)$$

where  $\theta$  is the angle between the direction of motion and the radius vector  $\mathbf{r}$  which has the components  $(x - Vt, y, z)$ . This expression – which is most easily obtained by transforming the Coulomb field from the rest frame of the charge to a moving frame using the Lorentz transformation properties of the electromagnetic fields – is more complicated but still possesses two key properties of the static charge. It falls as  $(1/r^2)$  at large distances *and it is radially directed from the instantaneous position* of the charge.

The energy flow corresponding to the electromagnetic field scales as the square of the electromagnetic field. If the field falls as  $(1/r^2)$ , the energy flux will fall as  $(1/r^4)$  and, since the area of a spherical surface scales as  $r^2$ , the total energy flowing through a sphere at large distances



**T Padmanabhan works at IUCAA, Pune and is interested in all areas of theoretical physics, especially those which have something to do with gravity.**

#### Keywords

Electromagnetism, theory of relativity, radiation.

When a charge is accelerating, something dramatic happens.

from the charge falls as  $r^2 \times (1/r^4) = (1/r^2)$ . Therefore, one cannot transfer energy to large distances in this kind of field. This is understandable because such a transfer cannot take place in the rest frame of the charge – in which we only have a static Coulomb field – and since we expect such a physical process to be Lorentz invariant it should not happen for a charge moving with uniform velocity either.

But when the charge is accelerating, something dramatic happens. The electric field, say, picks up an additional term which falls only as  $(1/r)$  at large distances. The change from the  $(1/r^2)$  dependence to the  $(1/r)$  dependence makes tremendous difference (and much of modern technology owes its existence to this fact). When the field falls as  $(1/r)$  at large distances, the energy flux will fall as  $(1/r^2)$  and the total energy flowing through a sphere at large distances from the charge,  $r^2 \times (1/r^2)$ , is a constant! Therefore, the fields arising from an accelerated charge are capable of transmitting energy to large distances from the charge. Clearly, it would be nice to understand better how acceleration leads to such a shift from  $(1/r^2)$  to  $(1/r)$  dependence – which changes the caterpillar to a butterfly.

In the case of an accelerated motion, the electric field picks up a transverse component which is perpendicular to the radial direction.

There is also another peculiar feature that arises when the charge undergoes an accelerated motion. The Coulomb field of a charge at rest, and that of a charge moving with a uniform velocity, is radial. The electric field vector in these cases points radially outward from the charge. But in the case of an accelerated motion, the electric field picks up a transverse component which is perpendicular to the radial direction. Since a propagating electromagnetic plane wave, for example, will have an electric field that is transverse to the direction of propagation of the wave, this fact is crucial for identifying the field generated by the acceleration with electromagnetic radiation.



It turns out that there is a remarkably elegant and simple way of understanding both these features, originally due to J J Thomson [1], which deserves to be known better and possibly could replace the unimaginative derivation using Lienard–Wiechert potentials in the classrooms!. (This derivation is discussed, for example, in [2] and also appears in the standard textbooks [3, 4] though in these textbooks an impression is created that the result is valid only for non-relativistic motion.) I will describe this approach and its essential features.

To begin with, let us consider a few elementary facts about Maxwell’s equations which connect the electromagnetic fields to the motion of the source. Since the electric field is  $\mathbf{E} = -(1/c)(\partial\mathbf{A}/\partial t) - \nabla\phi$ , we see that the electric field has a component which depends linearly on  $(\partial\mathbf{A}/\partial t)$ . It is also well known that the source for the vector potential  $\mathbf{A}$  is the current  $\mathbf{j}$  in the sense that  $\square\mathbf{A} \propto \mathbf{j}$ . Therefore  $(\partial\mathbf{A}/\partial t)$  will have a source that depends on  $(\partial\mathbf{j}/\partial t)$ . Since  $\mathbf{j}$  is linear in the velocity of the charge, we conclude that the electric field will have a source term which is linear in the time derivative of the velocity, viz., the acceleration  $\mathbf{a}$ .

An alternative way of understanding this result is as follows: A charge  $q$  moving with uniform velocity  $\mathbf{v}$  is equivalent to a current  $\mathbf{j} = q\mathbf{v}$ . This current will produce a magnetic field (in addition to the electric field) which scales in proportion to  $\mathbf{j}$ . If  $\mathbf{a} = \dot{\mathbf{v}} \neq 0$ , it will produce a nonzero  $(\partial\mathbf{j}/\partial t)$  and hence a nonzero  $(\partial\mathbf{B}/\partial t)$ . Through Faraday’s law, the  $(\partial\mathbf{B}/\partial t)$  will induce an electric field which scales as  $(\partial\mathbf{j}/\partial t)$ . (That is, if  $(\partial\mathbf{j}/\partial t)$  changes by factor 2, the electric field will change by factor 2.) It follows that an accelerated charge will produce an electric field which is linear in  $(\partial\mathbf{j}/\partial t) = q\mathbf{a}$ . (This, of course, is in addition to the usual Coulomb term which is independent of  $\mathbf{a}$  and falls as  $r^{-2}$ .)

Further, since the wave equation  $\square\mathbf{A} \propto \mathbf{j}$  propagates

Since  $\mathbf{j}$  is linear in the velocity of the charge, we conclude that the electric field will have a source term which is linear in the time derivative of the velocity, viz., the acceleration  $\mathbf{a}$ .



Before we do more sophisticated mathematics, let us try a bit of dimensional analysis to determine the electric field which arises from the acceleration.

information at the speed of light, we also know that the electric field at an event  $(t, \mathbf{x})$  is determined entirely by the behaviour of the source at the event  $(t_R, \mathbf{x}')$ , where  $t - t_R = (1/c)|\mathbf{x} - \mathbf{x}'| \equiv (r/c)$ . It is usual to call  $t_R$  the ‘retarded time’.

Before we do more sophisticated mathematics, let us try a bit of dimensional analysis to determine the electric field which arises from the acceleration. We know that the electric field has to be determined by the charge of the particle  $q$ , velocity of light  $c$ , acceleration  $a$  and the distance  $r$  (with  $a$  and  $r$  calculated at the retarded time). In general, the field will also depend on the velocity of the particle at the retarded time but we will choose a Lorentz frame in which the charge was at rest *at the retarded time* thereby eliminating any  $v$  dependence. We next use the fact that the electric field, which is linear in  $\partial\mathbf{j}/\partial t$ , should be linear in both  $q$  and  $a$  to write:

$$E = C(\theta) \frac{qa}{c^n r^m} = C(\theta) \left(\frac{q}{r^2}\right) \left(\frac{a}{c^n r^{m-2}}\right), \quad (2)$$

where  $C$  is a dimensionless factor, depending only on the angle  $\theta$  between  $\mathbf{r}$  and  $\mathbf{a}$ , and  $n$  and  $m$  need to be determined. (Since  $\mathbf{v} = 0$  in the instantaneous rest frame, the field cannot depend on the velocity.) From dimensional analysis, noting that  $E$  has the dimensions of  $q/r^2$  (Gaussian units, sorry!) it immediately follows that  $(a/c^n r^{m-2})$  must be dimensionless, leading to  $n = 2, m = 1$ . So we get the result:

$$E = C(\theta) \frac{qa}{c^2 r}. \quad (3)$$

Dimensional analysis plus the fact that  $\mathbf{E}$  must be linear in  $q$  and  $a$ , implies the  $r^{-1}$  dependence for the radiation term.

Thus, dimensional analysis plus the fact that  $\mathbf{E}$  must be linear in  $q$  and  $a$ , implies the  $r^{-1}$  dependence for the radiation term.

While this result shows why a term linear in acceleration will also have a  $(1/r)$  dependence, it does not really tell us how exactly it comes about. Moreover, dimensional



analysis cannot determine the nature of the dimensionless function  $C(\theta)$ . The argument due to J J Thomson [1] does both of these in an elegant way and I will describe a slightly modified version of the same.

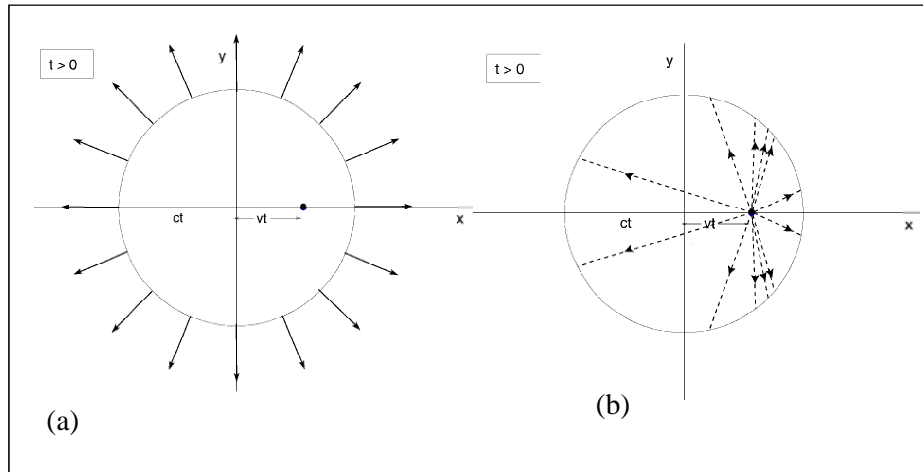
Let us consider a charged particle  $\mathcal{A}$  moving along some arbitrary trajectory  $\mathbf{z}(t)$ . We are interested in the electric field, say, produced at an event  $\mathcal{P}(t, \mathbf{x})$  by this charge. Since the characteristics of the wave equation shows that information propagates at the speed of light from the source point to the field point, we already know that the field at  $\mathcal{P}$  will be determined by the properties of the trajectory at the retarded time  $t_R$ . Further, the electric field can only depend on the position  $\mathbf{z}(t_R)$ , velocity  $\dot{\mathbf{z}}(t_R)$  and the acceleration  $\ddot{\mathbf{z}}(t_R)$  at the retarded time but not on higher time derivatives. (This follows from the fact that the source for electromagnetic field only involves up to the first time derivative of the current which will be proportional to the acceleration.) We will now choose our Lorentz frame such that the charge was at rest at the origin of the spacetime coordinates at the retarded time  $t_R = 0$ . Let the acceleration of the charge be  $\mathbf{a} = \ddot{\mathbf{z}}(t_R)$  at this instant. We will rotate the coordinate system so that  $\mathbf{a}$  is along the  $x$ -axis.

We now consider *another* charged particle  $\mathcal{B}$  which was at rest, at the origin, from  $t = -\infty$  to  $t = 0$  and undergoes constant acceleration  $\mathbf{a}$  along the  $x$ -axis for a short time  $\Delta t$ . For  $t > \Delta t$ , it moves with constant velocity  $v = a\Delta t$  along the  $x$ -axis. Let us study the electric field produced by this charge  $\mathcal{B}$  at some time  $t \gg \Delta t$ . Since  $\Delta t$  is arbitrarily small, we have  $a\Delta t \ll c$  and we can use the non-relativistic approximation throughout. *Since the trajectory of this charge matches identically in position, velocity and acceleration with the trajectory of the charged particle we are originally interested in, it follows that both of them will produce identical electric fields at  $\mathcal{P}$ .* This was the key insight of Thomson. As we shall see, the field produced by  $\mathcal{B}$  is fairly trivial to

Since the trajectory of this charge matches identically in position, velocity and acceleration with the trajectory of the charged particle we are originally interested in, it follows that both of them will produce identical electric fields at  $\mathcal{P}$ .



Figure 1.



calculate and hence we can obtain the field due to  $\mathcal{A}$ .

The ‘news’, that the charge was accelerated at  $t = 0$ , could have only travelled to a distance  $r = ct$  in time  $t$ . Thus, at  $r > ct$ , the electric field should be that due to a charge located at the origin as shown in *Figure 1a*:

$$\mathbf{E} = \frac{q}{r^2} \hat{\mathbf{r}} \quad (\text{for } r > ct). \quad (4)$$

At  $r \lesssim ct$ , the field is that due to a charge moving with velocity  $v$  along the  $x$ -axis, given by (1). The key point is that this field is radially directed from the instantaneous position of the charged particle. When  $v \ll c$ , which is the situation we are interested in, this is again a Coulomb field radially directed from the *instantaneous* position of the charged particle (see *Figure 1b*) :

$$\mathbf{E} = \frac{q}{r'^2} \hat{\mathbf{r}}' \quad (\text{for } r < ct). \quad (5)$$

The key point is that this field is radially directed from the instantaneous position of the charged particle.

Around  $r = ct$ , there exists a small shell of thickness  $(c\Delta t)$  in which neither result holds good. It is clear that the electric field in the transition region should interpolate between the two Coulomb fields. The crucial question is how to do this making sure that the flux of the



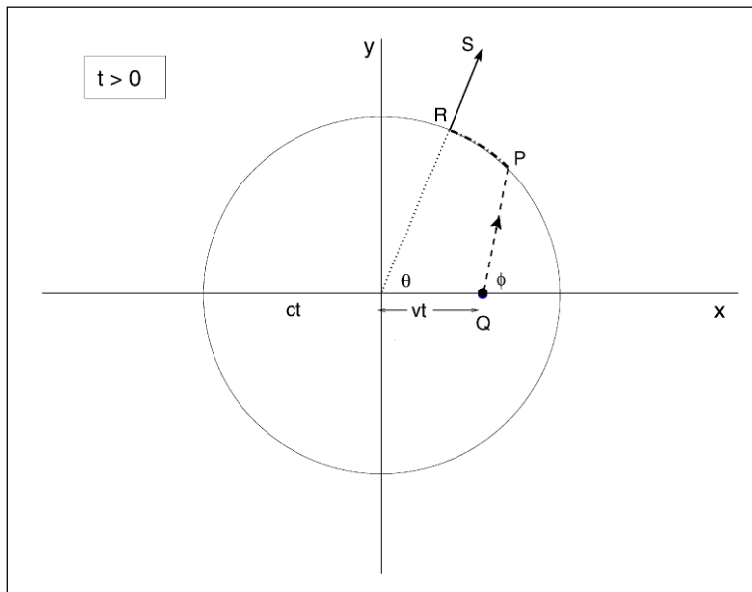


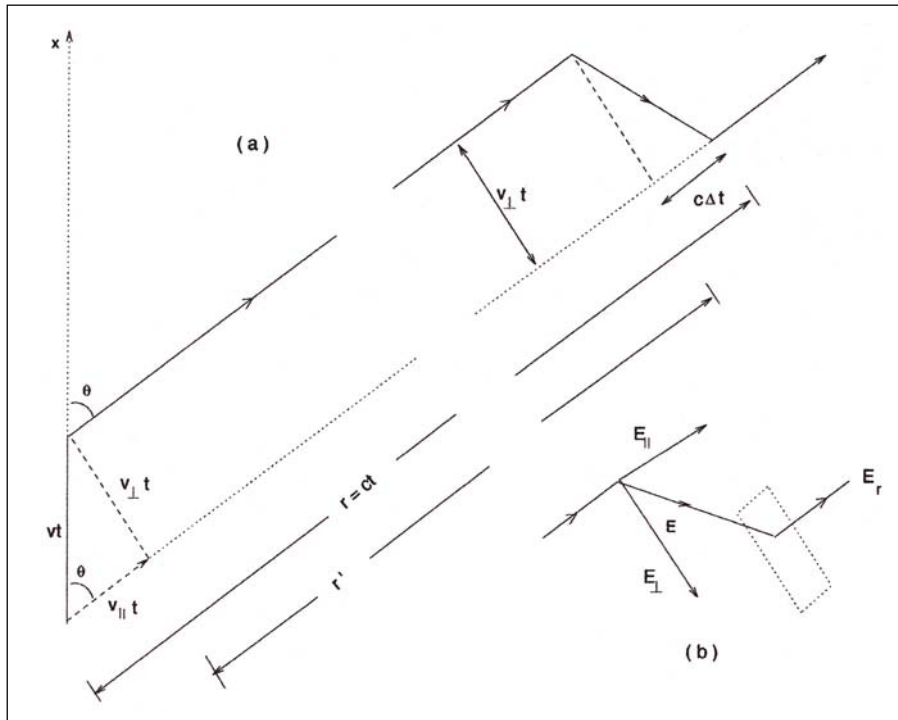
Figure 2.

electric field vector through any small box in this region vanishes, as it should to satisfy Maxwell's equations. As we shall see below, it turns out that this requires the field lines to appear somewhat as shown in *Figure 2*. One can explicitly work out this condition and prove that  $\tan \theta = \gamma \tan \phi$ , where  $\gamma = (1 - v^2/c^2)^{-1/2}$ . (You should try this out; it is done in detail in [3].) In the non-relativistic limit that we are considering,  $\theta \approx \phi$  making the field lines parallel to each other in the inside and outside regions; that is, QP is parallel to RS. (This is easy to understand because the radial field is just the Coulomb field both in the outside and in the inside region. For the flux to be conserved, these two field lines should be parallel to each other.) What is really interesting is that we now need a piece of electric field line PR interpolating between the two Coulomb fields. This is clearly transverse to the radial direction and all that we need to do is to prove that its magnitude varies as  $1/r$ . Let us see how this comes about.

The situation is described in detail in *Figure 3* which is self-explanatory. Let  $E_{\parallel}$  and  $E_{\perp}$  be the magnitudes of

What is really interesting is that we now need a piece of electric field line PR interpolating between the two Coulomb fields.





**Figure 3.**

**(a) The electric field due to a charged particle which was accelerated for a small time interval  $\Delta t$ . For  $t > \Delta t$ , the particle is moving with a uniform non-relativistic velocity  $v$  along the  $x$ -axis. At  $r > ct$ , the field is that of a charge at rest in the origin. At  $r < c(t - \Delta t)$ , the field is directed towards the instantaneous position of the particle. The radiation field connects these two Coulomb fields in a small region of thickness  $c\Delta t$ .**

**(b) Pill box construction to relate the normal component of the electric field around the radiation zone.**

the electric field parallel and perpendicular to the direction  $\hat{r}$ . From the geometry, we have

$$\frac{E_{\perp}}{E_{\parallel}} = \frac{v_{\perp} t}{c\Delta t} \tag{6}$$

But  $v_{\perp} = a_{\perp} \Delta t$  and  $t = (r/c)$  giving

$$\frac{E_{\perp}}{E_{\parallel}} = \frac{(a_{\perp} \Delta t) (r/c)}{c\Delta t} = a_{\perp} \left( \frac{r}{c^2} \right) \tag{7}$$

The value of  $E_{\parallel}$  can be determined by using Gauss theorem to a small pill box, as shown in the small inset in Figure 3. This gives  $E_{\parallel} = E_r = (q/r^2)$ ; thus we find that

$$E_{\perp} = a_{\perp} \left( \frac{r}{c^2} \right) \cdot \frac{q}{r^2} = \frac{q}{c^2} \left( \frac{a_{\perp}}{r} \right) \tag{8}$$





This is the radiation field located in a shell at  $r = ct$ , which is propagating outward with a velocity  $c$ . The above argument clearly shows that the origin of the  $r^{-1}$  dependence lies in the necessity to interpolate between the two Coulomb fields. We have thus determined the electric field generated due to the acceleration of the charge and have shown that it is transverse and also falls as  $(1/r)!$

We can express this result more concisely in the vector notation as

$$\mathbf{E}_{\text{rad}}(t, \mathbf{r}) = \frac{1}{c^2} \left[ \frac{1}{r} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}) \right]_{\text{ret}}, \quad (9)$$

where  $\hat{\mathbf{n}} = (\mathbf{r}/r)$  and the subscript ‘ret’ implies that the expression in square brackets should be evaluated at  $t' = t - r/c$ . Comparison with equation (3) shows that  $C(\theta) = \sin\theta$ . The full electric field *in the frame in which the charge is instantaneously at rest* is  $\mathbf{E} = \mathbf{E}_{\text{coul}} + \mathbf{E}_{\text{rad}}$ . We emphasise that this result is exact in the Lorentz frame in which the charge was at rest at the retarded time. (One does not have to make a non-relativistic ‘approximation’ because  $v = 0$  automatically takes care of it!). If we now make a Lorentz transformation to a frame in which the particle was moving with some velocity  $\mathbf{v} = \dot{\mathbf{z}}(t_R)$  at the retarded time, then we can obtain the standard, fully relativistic, expression with the velocity dependence. This is algebraically a little complicated because one needs to make a Lorentz transformation in an arbitrary direction since  $\mathbf{v}$  and  $\mathbf{a}$  will not, in general, be in the same direction. (This is done in [5] if you are interested.) Thus J J Thomson’s idea is quite capable of giving us the complete solution to the problem.

### Suggested Reading

- [1] J J Thomson, *Electricity and Matter*, Archibald Constable, London, Chapter III, 1907.
- [2] T Padmanabhan, *Cosmology and Astrophysics through Problems*, Cambridge University Press, 1996.
- [3] E M Purcell, *Electricity and Magnetism*, The Berkeley Physics Course, Vol.2, 2nd ed. McGraw-Hill, New York, 2008.
- [4] F S Crawford, *Waves*, The Berkeley Physics Course, Vol.3, 2nd ed. McGraw-Hill, New York, 2008.
- [5] Hamsa Padmanabhan, A simple derivation of the electromagnetic field of an arbitrarily moving charge, *Am. J. Phys.*, Vol.77, pp.151–155, 2009, [arXiv:0810.4246]

#### Address for Correspondence

T Padmanabhan  
 IUCAA, Post Bag 4  
 Pune University Campus  
 Ganeshkhind  
 Pune 411 007, India  
 Email:  
 paddy@iucaa.ernet.in  
 nabhan@iucaa.ernet.in

