Think It Over

This section of Resonance presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to ‘Think It Over’, Resonance, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

Solution to ‘Are you Better off than Your Friend?’

There are two friends A1 and A2 and two boxes B1 and B2. One box has twice as much money as the other one. Neither A1 nor A2 knows which one has more money. The friends decide that they will toss a fair coin. If it is heads A1 will get B1 and A2 will get B2 and if it is tails A1 will get B2 and A2 will get B1.

After the boxes are selected but before they are opened A1 suggests to A2 that they should exchange the boxes as it is “good” for both of them in the sense that the ratio of the amount after exchange to before exchange has an average value greater than one. Is A1 right?

Yes. A1 is correct.

Let \(X_1\) denote A1’s money before the exchange and \(X_2\) denote the same for A2. Then, for A1 the ratio of the money after exchange to before exchange is \(Y_1 = \frac{X_2}{X_1}\) and that for A2 is \(Y_2 = \frac{X_1}{X_2}\). The claim made by A1 is that the average value of \(Y_1\) is greater than 1 and the same is true for \(Y_2\). To see this, note that the random vector \((X_1, X_2)\) is either \((c,2c)\) or \((2c,c)\) with probability 1/2 each, for some \(c > 0\). Thus \(Y_1\) is 1/2 or 2 with probability 1/2 each. Thus the average value of \(Y_1\) (also called the mean value of \(Y_1\) or the expected value of \(Y_1\) and written \(EY_1\)) is \(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 2 = \frac{5}{4}\) which is indeed greater...
than one. The same argument shows that the average value of $Y_2$ is also $\frac{5}{4}$.

**Generalisation**

Now see if you can establish the following more general result. Let $X_1$ and $X_2$ be two random quantities taking finitely many values such that (i) both are always greater than zero; (ii) the random vectors $(X_1, X_2)$ satisfy $\Pr(X_1 = x, X_2 = y) = P(X_1 = y, X_2 = x)$ for all $x, y$. Show that $E(X_1/X_2) = E(X_2/X_1) \geq 1$ with ‘$=$’ holding iff $X_1/X_2$ is not random.

(Hint: Show and use the fact that $x + 1/x$ is $\geq 2$ for all $0 < x < \infty$).

“I dropped in to tell you that I have found the link between your factors and chromosomes!”