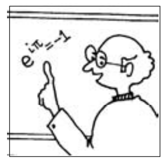

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

The Simple Pendulum: Not So Simple After All!

H P Kaumudi
BSc Final Year
St. Joseph's College
Bangalore 560 027, India.

Vasant Natarajan
Department of Physics
Indian Institute of Science
Bangalore 560 012, India.
Email:profvasant@gmail.com

Theoretical analysis of the 'simple' pendulum beyond the small-angle approximation teaches us about anharmonic oscillators and approximate (perturbative) solutions. Experiments with a simple-to-set-up pendulum teach us about errors in measurements, and curve fitting of theoretical expressions to experimental data. The pendulum also exhibits frictional damping, a feature of most real oscillators.

1. Introduction

Galileo Galilei (1564–1642) first introduced the concept of a simple pendulum when he was watching a lamp in a church oscillate (it was his responsibility to see that it kept on burning). He was a medical student at that time, so he used his pulse to record the time taken for the oscillations. He found that, contrary to common belief, the time taken by the pendulum for each oscillation is the same independent of the amplitude. Nowadays, we study the simple pendulum in high school and learn

Keywords
Harmonic oscillator, anharmonicity, damping, measurement errors, curve fitting.

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that this independence of time period with amplitude is the essence of a harmonic oscillator. But the pendulum behaves in this 'simple' manner only when we make the so-called small-angle approximation. At larger angles, this approximation breaks down and the motion of the pendulum shows interesting new physics.

In this article, we will see how the solution changes when we have to go beyond the small-angle approximation, making the oscillator 'anharmonic'. We will first see how to theoretically solve the anharmonic equation of motion using approximation methods. We will then set up an experiment to verify these results. The same experiment will show us another feature of real-world oscillators, namely frictional damping. Analysis of the experimental data will teach us about measurement errors and curve fitting to theoretical expressions.

2. Theoretical Analysis

The parameters defining the pendulum are its length, l, and the bob mass, m, as shown in *Figure* 1. The motion of the pendulum is parametrized by the angle θ, and our problem is to find the time evolution of this parameter. To solve this, it is most convenient to start with Newton's second law of motion, $F = ma$, but modified as below for angular displacements:

$$N = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\,, \tag{1}$$

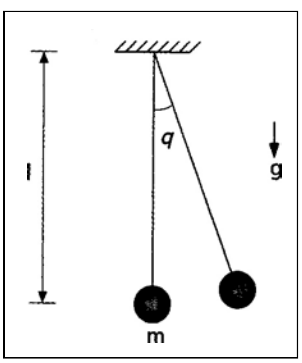
where $N = -mgl\sin\theta$ is the *restoring torque*, and $I = ml^2$ is the moment of inertia, both defined about the point of support. Substituting in (1), the equation of motion becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0\,. \tag{2}$$

If we now make the small-angle approximation that $\sin\theta \approx \theta$, we get the familiar harmonic oscillator equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2\theta = 0 \tag{3}$$

Figure 1. The simple pendulum showing the relevant parameters.



with the angular frequency defined as $\omega_0 = \sqrt{g/l}$. The solution is

$$\theta(t) = \theta_0 \cos(\omega_0 t + \phi_0)\,, \tag{4}$$

with the amplitude θ_0 and phase ϕ_0 obtained from the initial (or boundary) conditions. The important point about the solution is that the frequency is independent of the amplitude[1], the very definition of harmonic motion.

[1]It is interesting to note that the frequency is also independent of *the mass of the bob*. This is a consequence of the *equivalence principle*, also first discovered by Galileo, which states that the inertial and gravitational masses of an object are the same, or that all objects fall with the same acceleration under gravity. This forms an important input into Einstein's theory of gravity, the General Theory of Relativity.

Now let us see what happens if we do not make the small-angle approximation for the restoring torque. A Taylor series expansion for the sine function around $\theta = 0$ yields,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\,. \tag{5}$$

In *Figure* 2, we compare the restoring torque with the proper sinusoidal dependence, against the curves if we make the small-angle approximation (linear restoring torque), and if we include the next term in the Taylor series. Just the addition of the θ^3 term makes the curve almost identical to the sine curve up to angles of about 1 rad (57°). So we can get a much better description of the motion for large angles by including this term in the

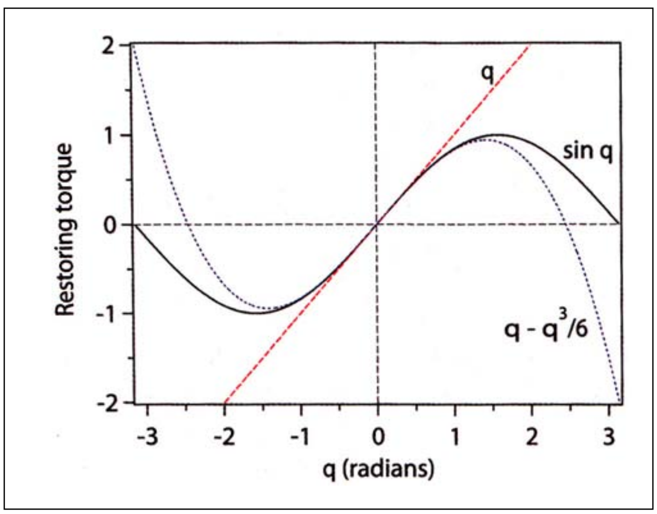


Figure 2. Comparison of the restoring torque with the small-angle approximation (dashed), with the next term in the Taylor series (dotted), and the actual sinusoidal curve (solid). For simplicity, mgl is taken to be 1.

The restoring torque is smaller than that with the linear approximation, therefore we expect that the time period of oscillation will increase as the amplitude increases.

equation of motion. Before we proceed, let us see what to expect physically because of the addition of this term. The curves in *Figure* 2 show us that the restoring torque is smaller than that with the linear approximation, therefore we expect that the time period of oscillation will increase as the amplitude increases.

Let us now see how this comes about mathematically. With the next term in the Taylor series, the equation of motion we are trying to solve is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2\theta - \frac{\omega_0^2}{6}\theta^3 = 0\,. \tag{6}$$

We first note that the extra term is small compared to the first term. Thus, we can get an approximate (or what is called first-order) solution to the above equation by guessing that the motion is still sinusoidal, but with a frequency that is now amplitude *dependent*, i.e.,

$$\theta(t) = \theta_0 \cos(\omega t)\,, \text{with} \;\; \omega = \omega(\theta_0) \neq \omega_0.$$

Substituting our guess solution into the equation of motion gives a term like $\cos^3(\omega t)$. We use the trigonometric identity

$$4\cos^3(\omega t) = \cos(3\omega t) + 3\cos(\omega t)\,.$$

The approximation in the solution, called the 'secular' approximation, now comes if we neglect the fast-varying $\cos(3\omega t)$ term. The justification for this is that the term varies three times as rapidly as the other $\cos(\omega t)$ terms, and therefore its contribution will average to zero over a cycle. With this approximation, (6) yields,

$$\omega^2 = \omega_0^2\left(1 - \frac{\theta_0^2}{8}\right) \approx \omega_0^2\left(1 + \frac{\theta_0^2}{8}\right)^{-1}, \tag{7}$$

The 'secular' approximation, now comes if we neglect the fast-varying $\cos(3\omega t)$ term.

where we have used the binomial expansion to get the second term. Thus, in terms of time period,

$$T = T_0\sqrt{\left(1 + \frac{\theta_0^2}{8}\right)}\,, \tag{8}$$

with $T_0 = 2\pi/\omega_0$. This shows that the frequency is amplitude dependent and the time period increases with amplitude, exactly as expected from physical arguments.

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We have thus found this approximate solution by first noting that, since the modified equation of motion contains a small 'perturbation' to the original harmonic-oscillator equation, the solution will still be oscillatory, but with a perturbed frequency. Then, we made the 'secular' approximation of neglecting fast-varying terms. This is a general method that can be applied to solve many other problems where the perturbation is weak. A more detailed derivation of this result can be found in several textbooks ([1], for example).

The motion in the above analysis, though not harmonic, still remains *periodic*, i.e., the pendulum returns to every point in its trajectory because there is no mechanism by which it can lose energy. But real-world oscillators do lose energy – through frictional damping. This kind of damping is introduced into the equation of motion by adding a term proportional to the (angular) velocity:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega_0^2\theta = 0\,. \tag{9}$$

For simplicity, we have ignored the anharmonic part. One can easily verify that the solution is of the form

$$\theta(t) = \theta_0 \exp(-t/\tau)\cos(\omega' t)\,, \tag{10}$$

i.e., an amplitude that decays exponentially as the oscillator loses energy, and a frequency that is slightly modified from the unperturbed frequency. The damping time constant and the modified frequency are given by:

The pendulum returns to every point in its trajectory because there is no mechanism by which it can lose energy.

$$\tau = \frac{2m}{b} \quad \text{and} \quad \omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,.$$

For weak damping, i.e. $b/2m \ll \omega_0$, the frequency is close to the unperturbed value, and only the amplitude decays slowly over several oscillations.

The use of a checknut to attach the string ensures an almost ideal point contact.

Let us now verify these results experimentally.

3. Experimental Details

The apparatus required for the experiment are: A lead bob, a string of approximately 1 m length, a board, a stand, a checknut, and a digital timer. On the board, angles from 5° to 45° are marked at intervals of 1°. The bob is tied to the string so that the sum of the length of the string, the radius of the bob, and the length of the hook (i.e. the total length of the pendulum) is 1 m. This gives a calculated time period of $2\pi\sqrt{l/g} = 2.007$ seconds. The string is fixed to the stand with the board behind it to indicate the angle of the pendulum. The use of a checknut to attach the string ensures an almost ideal point contact.

The digital timer is the main piece of equipment used to eliminate human errors in measuring the time period of oscillations. It works in the following way. The device has a photosensor consisting of a light source and a detector. It is placed at the mean position of the pendulum such that the bob trips the detector every time it passes through this point. The timer is set such that it starts measuring the first instant it is tripped, and stops at the third instant it is tripped, by which time the bob has completed one full oscillation. The timer reading thus gives the time period for the oscillation.

The experimental procedure consisted of starting the pendulum from rest at a given angle (measured against the board), and measuring the time period for one oscillation. The measurement was repeated 10 times, and the average and standard deviation calculated for each set. This was done for angles varying from 5° to 45° in steps of 5°. From *Figure* 2, we know that 45° is a large enough angle to see the effects of anharmonicity.

For the damping measurement, a small starting angle of 20° was used so that the constancy of time period with amplitude is a reasonably good assumption. Then the total number of oscillations for reducing the amplitude by 2° was noted, i.e., the number of oscillations for reaching 19°, then 17°, then 15°, and so on. Since the time period per oscillation is constant, the number of oscillations for reaching a certain amplitude is directly related to the time to reach this amplitude, and the amplitude vs. number of oscillations should show an exponential decay according to equation (10).

If we do N measurements, then, assuming that the errors are distributed 'normally', the error in the average is the standard deviation of the N points divided by $\sqrt{N}$.

4. Results and Discussion

The results of the time period vs. angle are shown in *Figure* 3. Each point on the graph (solid circle) represents the average of 10 individual measurements. If we do N measurements, then, assuming that the errors are distributed 'normally', the error in the average is the standard deviation of the N points divided by $\sqrt{N}$. This is how the error in each measured point is evaluated. It is shown on the graph as a vertical 'error bar' on each point, which gives an idea of how much experimental uncertainty there is in that value.

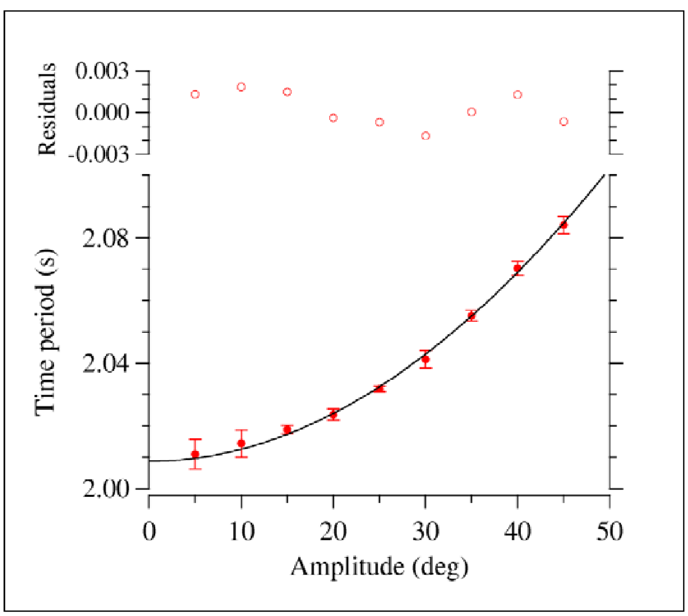

Figure 3. Increase of time period of oscillation with amplitude. The measured time period is plotted as a function of starting angle of the pendulum (filled circles). The solid curve is a fit to equation (8) with the fit residuals shown on top as open circles. The vertical error bars and fitting procedure are explained in the text.

T_0 is not an 'adjustable' parameter in our theory, but just a scale parameter that depends on the pendulum length and the local acceleration due to gravity.

The measured time period increases with amplitude, as we expected from our theoretical analysis. Now we have to see if the measured behaviour matches the theoretical prediction derived in (8). For this, we have to do a 'curve fit' to the expression, which means that we have to adjust a few fit parameters so that the calculated time period at each amplitude matches the measured one *as best as possible*. One way to mathematically define this term 'as best as possible' is to minimize the sum of the squares of the deviation between the measurement and calculation, the so-called least squares fitting. The solid curve in the graph is precisely this kind of least-squares fit, using the theoretical expression in (8). It matches the measured data quite well. The only fit parameter in the equation is T_0, and the best fit is obtained with a value of $T_0 = 2.0087 \pm 0.0006$ s, very close to our 'design' value of 2.007 s. Note that T_0 is not an 'adjustable' parameter in our theory, but just a scale parameter that depends on the pendulum length and the local acceleration due to gravity. A different pendulum at a different location will have a different T_0, but the dependence of T on amplitude will still be that given by (8). Thus any adjustment of T_0 will not change the shape of the curve in *Figure* 3, but only move it up or down. The fact that the experiment matches theory shows that we have analyzed the problem correctly and done the experiment without serious errors.

Finally, we expect the theory to match the measured values only within the experimental uncertainty. In other words, the fitted curve is expected to pass through not each point but only the vertical error bar around it. The residuals shown on top of the graph (open circles) give the difference between the measured value and the fitted value. These should be smaller than the error bars, and random if the errors are random (not 'systematic'). This is indeed what we observe.

Now we turn to the second measurement, namely damp-

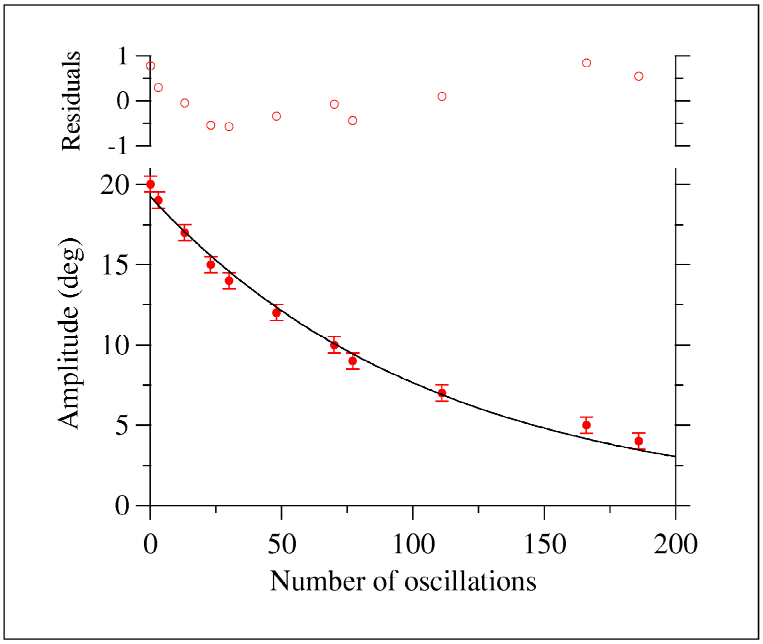

Figure 4. Decay of amplitude due to damping. The amplitude of the pendulum after a certain number of oscillations is shown as filled circles. The solid curve is a fit to (10) and the fit residuals are shown on top.

ing of the pendulum. In *Figure* 4, we show the amplitude as a function of the number of oscillations. The error here is determined not by taking the statistical average of many measurements, but by noting that the determination of the angle against the board is subject to error. Since the 'least count' of the markings is 1°, we can expect that the error will at most be ±0.5° around each angle. This is the size of the vertical error bars in the graph. As before, the solid curve is a fit to the exponential decay predicted by (10). In this case, the fit parameters are the initial amplitude and the damping constant. The best fit values are:

$$\theta_0 = 19.2 \pm 0.3 \quad \text{and} \quad n_\tau = 108 \pm 4\,,$$

which means that it takes about 108 oscillations for the amplitude to damp to $1/e$ of its starting value, showing that the damping is indeed weak. The fit residuals on top are of order 0.5°, as expected from our error bars, and are randomly distributed.

These kinds of simple experiments teach us about measurement errors and curve fitting to theoretical expressions.

5. Conclusions

In summary, we have seen that the 'simple' pendulum can teach us two important features of real-world oscillators, namely anharmonicity and damping. Theoretical analysis of the motion beyond the small-angle approximation (harmonic regime) shows that the time period increases with amplitude due to the reduced restoring torque. And the presence of friction, even in the harmonic regime, causes the amplitude to decay exponentially. A simple experimental set-up was used to verify these results. The measured time period up to a starting angle of 45° follows the theoretical prediction within the experimental uncertainty. For small-angle oscillations, the amplitude damps exponentially, exactly as expected. These kinds of simple experiments teach us about measurement errors and curve fitting to theoretical expressions.

Acknowledgements

H P K thanks H R Madhusudana of Bangalore Association for Science Education, Jawaharlal Nehru Planetarium, for help in conducting the experiments.

Suggested Reading

[1] **V V Migulin, V Medvedev, E Mustel, and V Parygn, *Basic Theory of Oscillations*, Mir Publishers, Moscow, 1983.**