George Andrews' Game

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This is an expository article showing how Zeckendorf's Theorem (every positive integer can be represented in one and only one way as the sum of non-consecutive Fibonacci numbers) can be used to construct a number-guessing game invented by Professor George Andrews.

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Introduction

The numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,$$
$$144, 233, 377, 610, 987, 1597, \ldots \tag{1}$$

are widely known as the *Fibonacci numbers*. They follow a simple rule: beginning with the third, each number is the sum of its two predecessors. The Fibonacci numbers were named after Leonardo of Pisa – known as Fibonacci – who wrote about them in his book *Liber Abaci*, published in 1202 CE. These numbers had been previously described by several Indian authors, as early as 600 CE [1]. They were used in the study of *prosody*, the patterns of rhythm and sound in poetry.

Research on Fibonacci numbers and their applications continues today. For example, *The Fibonacci Quarterly* is a journal devoted to research related to the Fibonacci numbers. In this article we describe a clever "number-guessing game" created by Professor George Andrews of Pennsylvania State University. The game depends upon an interesting property of the Fibonacci numbers, a property discovered by Edouard Zeckendorf in 1972. Before discussing Zeckendorf's Theorem and the game, we prove one preliminary result.

Keywords
Fibonacci numbers, Zeckendorf's theorem, Andrew's game, greedy algorithm.

Sums of Alternate Fibonacci Numbers

If we label the Fibonacci numbers in (1) as $F_1, F_2, F_3, \ldots$, we note that

$$F_2 = F_3 - 1\,.$$

Adding F_4 to both sides gives

$$F_2 + F_4 = F_3 + F_4 - 1 = F_5 - 1\,.$$

Adding F_6 to both sides of this result gives

$$F_2 + F_4 + F_6 = F_5 + F_6 - 1 = F_7 - 1\,.$$

This pattern continues leading to the following:

Lemma 1. *If $j \geq 2$, then*

$$\sum_{i=2}^{j} F_{2i-2} = F_{2j-1} - 1$$

and

$$\sum_{i=2}^{j} F_{2i-1} = F_{2j} - 1\,.$$

Proof. These results are easily proved by mathematical induction. □

Zeckendorf's Theorem

Edouard Zeckendorf (1901–1983) was a citizen of Belgium, a medical doctor, an army officer, a prisoner of war (1940–1945), and an amateur mathematician. He proved in [3] the result now called 'Zeckendorf's Theorem', that every positive integer $n \geq 2$ can be represented in one and only one way as the sum of nonconsecutive Fibonacci numbers. Such a sum is called the *Zeckendorf representation* of n.

As an example, the Zeckendorf representation of $n = 2000$ is

$$2000 = 1597 + 377 + 21 + 5\,.$$

Such representations can be found by using a 'greedy' algorithm. First, we choose the largest possible Fibonacci number less than or equal to 2000. Referring to (1), we find $F_{k_1} = 1597$. Next, we look for the largest Fibonacci number F_{k_2} less than or equal to $2000 - 1597 = 403$. We find $F_{k_2} = 377$. In the same way, $F_{k_3} = 21$ and $F_{k_4} = 5$.

The proof of Zeckendorf's Theorem given below is adapted from that given on pages 281–282 of [2]. We use their notation for 'non-consecutive':

$$j \gg k \iff j \geq k + 2\,.$$

Theorem (Zeckendorf's Theorem). *Every positive integer n has a unique representation of the form*

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \quad k_1 \gg k_2 \gg \cdots \gg k_r \gg 0\,. \tag{2}$$

Proof. We divide the proof into two parts. First, we show that if we apply the greedy algorithm to a positive integer n, it will produce a Zeckendorf representation for n and, second, if we have a Zeckendorf representation for n, then it is the same as that produced by the greedy algorithm.

Part 1 If n is a positive integer, the greedy algorithm chooses the largest Fibonacci number F_{k_1} less than or equal to n. This means that $F_{k_1} \leq n < F_{k_1+1}$. If it happens that $F_{k_1} = n$, then, with $r = 1$, (2) holds and the algorithm terminates. Otherwise, $0 < n - F_{k_1} < n$. We apply the greedy algorithm to $n - F_{k_1}$, thus finding a Fibonacci number F_{k_2} satisfying $F_{k_2} \leq n - F_{k_1} < F_{k_2+1}$. The algorithm continues in this way, terminating after a finite number of steps.

We show that $k_1 \gg k_2$ by showing that $n - F_{k_1} < F_{k_1-1}$. This inequality is true because to suppose otherwise, that is, $n - F_{k_1} \geq F_{k_1-1}$, leads to a contradiction. For,

$$n - F_{k_1} \geq F_{k_1-1} \quad \Longrightarrow \quad n \geq F_{k_1} + F_{k_1-1} = F_{k_1+1}\,.$$

This is a contradiction since by definition of k, $F_{k_1} \leq n < F_{k_1+1}$. The remaining inequalities in $k_1 \gg k_2 \gg \cdots \gg k_r \gg 0$ follow in the same way.

Part 2 For the second part of the proof, suppose that n is given by (2). We show that this representation is identical to that produced by the greedy algorithm. *Figure* 1 reminds us that larger subscripts correspond to larger Fibonacci numbers. Using the Lemma, if k_1 is even

$$\begin{aligned} n - F_{k_1} = F_{k_2} + F_{k_3} + \cdots + F_{k_r} &\leq F_{k_1-2} + F_{k_1-4} + \cdots + F_2 \\ &= F_{k_1-1} - 1\,; \end{aligned}$$

if k_1 is odd

$$\begin{aligned} n - F_{k_1} = F_{k_2} + F_{k_3} + \cdots + F_{k_r} &\leq F_{k_1-2} + F_{k_1-4} + \cdots + F_3 \\ &= F_{k_1-1} - 1\,. \end{aligned}$$

It follows from this that

$$0 \leq n - F_{k_1} \leq F_{k_1-1} - 1\,.$$

By adding F_{k_1} to the left side, middle, and right side of this inequality,

$$F_{k_1} \leq n \leq F_{k_1} + F_{k_1-1} - 1 = F_{k_1+1} - 1 < F_{k_1+1}\,.$$

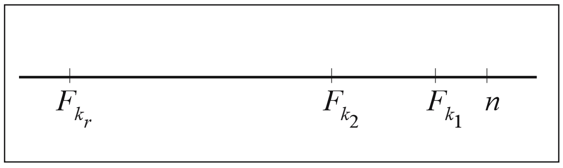

Figure 1. Recall that $k_1 \gg k_2 \gg k_3 \gg 0$.

This means that F_{k_1} is the largest Fibonacci number less than or equal to n. It follows that any representation of n having the form described in (2) is identical to the representation found by the greedy algorithm. □

George Andrews' Game

George E Andrews is the Evan Pugh Professor of Mathematics at Pennsylvania State University. Professor Andrews's research is in number theory. While on a trip to Europe in 1957, Andrews discovered the *Lost Notebook* of the great Indian mathematician Srīnivāsa Rāmāṇujaṇ. Since that time, Andrews, several colleagues, and his students have been studying the many assertions contained in this *Notebook*.

During a lecture at Iowa State University, Ames, Iowa, USA in 2008, Andrews presented his number-guessing game. The game is based on the uniqueness of the Zeckendorf representations of integers. In more or less his own words, this is how the game is played: Professor Andrews asks for a volunteer from the audience. He asks this assistant to choose a positive integer between 1 and 50, to write this integer on a piece of paper, and then to show it to the audience, all done while Andrews is looking away. Andrews then says to the audience, "This is an interactive trick. I will ask my assistant whether his/her number is on one or more of a series of eight cards. If the number is not on a card, then we shall set the card aside as it is of no interest to either of us. If the number is on the card, then it will be my job to say to him/her whether his/her number is on the next card. Then, whenever the assistant answers 'yes,' I hold up the next card, scratch my head, appear to be deep in thought, and eventually say, 'No, your number is not on this card!'" Eventually, Professor Andrews identifies the chosen number and announces it to the audience. Two of the cards are shown in *Figure* 2.

1	4	6	9	12
14	17	19	22	25
27	30	33	35	38
40	43	46	48	51
53	56	59	61	64

8	9	10	11	12
29	30	31	32	33
42	43	44	45	46
63	64	65	66	67
84	85	86	87	88

Figure 2.

How does this "trick" work? The Zeckendorf representations of all positive integers less than or equal to 50 require the eight Fibonacci numbers $1, 2, 3, 5, 8, 13, 21, 34$. If we are able to infer the representation of the chosen number, then we can determine that number by adding the numbers in its Zeckendorf representation.

The Cards

On the ith card we place the ith number from $1, 2, 3, 5, 8, 13, 21, 34$ in the upper-left-corner. We fill the remaining 24 positions with the numbers from 1 to 50 whose Zeckendorf representations include the given ith number and, to avoid duplicates, numbers larger than 50 whose Zeckendorf representations include the given ith number. We show the cards in the natural order of the upper-left-corner numbers.

If a card produces a "yes," then the next card can be skipped because consecutive Fibonacci numbers do not occur in the representation of the chosen number. To find the chosen number requires only that Professor Andrews add up the Fibonacci numbers corresponding to the "yes" answers.

The remaining six of the eight cards required for the '1–50' version of the game are shown in *Figure* 3. It is not difficult to write computer programs to generate a list of Fibonacci numbers, to find the Zeckendorff representation of a given positive integer, and, for the ith Fibonacci number, to select numbers for that card.

Figure 3.

2	7	10	15	20
23	28	31	36	41
44	49	54	57	62
65	70	75	78	83
86	91	96	99	104

3	4	11	12	16
17	24	25	32	33
37	38	45	46	50
51	58	59	66	67
71	72	79	80	87

5	6	7	18	19
20	26	27	28	39
40	41	52	53	54
60	61	62	73	74
75	81	82	83	94

13	14	15	16	17
18	19	20	47	48
49	50	51	52	53
54	68	69	70	71
72	73	74	75	102

21	22	23	24	25
26	27	28	29	30
31	32	33	76	77
78	79	80	81	82
83	84	85	86	87

34	35	36	37	38
39	40	41	42	43
44	45	46	47	48
49	50	51	52	53
54	123	124	125	126

Suggested Reading

[1] **Parmanand Singh, The So-called Fibnonacci Numbers in Ancient and Medieval India, *Historia Mathematica*,Vol.12, pp.229–244, 1985.**

[2] **Ronald L Graham, Donald E Knuth, and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley Publishing Company, Reading, Massachusetts, October 1990.**

[3] **E Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, B*ulletin de la Société Royale des Sciences de Liége*,Vol.41, pp.179–182, 1972.**

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