Mathematics of Risk Taking

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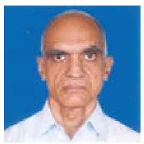

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A simple mathematical model for an investor's gains and losses over time shows that, in the long run, those with large sums to invest have an excellent chance of reaching their goal while the marginal investors have a high probability of going bankrupt. A greedy investor, rich or poor, will hit the bottom in the long run, with probability one. Consequences for the population at large are discussed.

1. Introduction

Is gambling good for you ? The answer is, of course, No. Not just for moral or ethical reasons, since gambling is a perverse way of risk taking, but for good practical reasons based on some sound and simple mathematics.

Mathematics, like philosopher's God, is impersonal, and the mathematics of risk taking is independent of who is taking risk and for what purpose. So the article is also about the genuine and needed risk taking of entrepreneurs (which includes farmers), businessmen and investors.

Our main results are in Sections 2 and 3. The implications for the population at large are discussed in Section 4.

2. A Simple Gambling Scheme (also known as Simple Symmetric Random Walk)

Suppose the gambling scheme consists of tossing a fair coin, (i.e., the probability of the coin falling heads is 1/2, which is also the probability of tails). At each toss the gambler receives one rupee from the gambling house if the coin falls heads up, and she gives one rupee to the gambling house if it falls tails up. Suppose the gambler starts with an initial capital of Rs.$a > 0$, and let us suppose that she wishes to increase it to Rs.$N > a$ and that she stops gambling as soon as the goal N reached. In case the gambler reaches 0, before reaching N, then too the game stops since the gambler has no money left to gamble.

What do you think will happen ?

Clearly there are three events possible:

1) The tossing of the coin continues indefinitely, i.e., the gambler never accomplishes the goal of reaching N, nor does she become bankrupt by reaching 0.

2) The gambler accomplishes the goal of reaching N in a finite number of steps (before reaching 0) and stops gambling any further.

3) The gambler becomes bankrupt, i.e., reaches 0 in a finite number of tosses (before reaching N) so the game stops.

The following proposition is a consequence of more general considerations of the next section (which includes the above gambling scheme as a special case).

Proposition 2.1 *(i) The probability of the first event is zero or equivalently, the gambling will almost surely end in a finite number of tosses,*

(ii) the probability of event (2) of the gambler reaching N in a finite number of tosses (before reaching 0, i.e., becoming bankrupt) is $\frac{a}{N}$, where a is the initial capital of the gambler,

(iii) the probability of event (3) of the gambler becoming bankrupt in finite number of tosses is $1 - \frac{a}{N}$.

Thus, for example, if $N = 10^7$ (a crore), the gambler's

Simple Random Walk

At time zero start at some integer a. Then toss a coin whose probability for 'heads' is p , $0 < p < 1$. If 'heads', move to the right one step, i.e., to $(a + 1)$ and if 'tails', move to the left one step , i.e., to $(a – 1)$. This is your position at time one. Now repeat this at each time step with the tosses being independent. The trajectory you generate is called a simple random walk with initial value a and parameter p. If $p = 1/2$ then it is called simple symmetric random walk.

This model is of relevance in many areas of science such as physics, operations research, mathematical finance and population genetics.

chances of becoming a crorepati are high, say 0.7 or more if her initial capital a is 70 lakhs or more. On the other hand if her capital is 3 lakhs or less, her chances of becoming a crorepati is only 0.03 or less. Thus a rich gambler has a higher probability of reaching her goal than a poor one.

The above gambling scheme may also be viewed as a kind of walk, called simple symmetric random walk with absorbing barriers. Consider a person starting at a positive integer a, $0 < a < N$, and executing a walk along the x-axis as follows: she tosses a fair coin and takes a step of unit length in positive direction if the coin falls heads up, and she takes a step of unit length in the negative direction if the coin falls tails up. The process is repeated at the person's new position, and so on. The walk stops as soon as the person reaches 0 or N, these being the absorbing barriers. Proposition 2.1 says that the walk will end in a finite number of steps with probability one and that the probabilities of reaching N and 0 are $\frac{a}{N}$ and $1 - \frac{a}{N}$ respectively.

Let X_i denote the gambler's possible capital at time i. It is a random variable. Assume that the game has not stopped up to the nth toss and that $a_1, a_2, \cdots, a_n$ are respectively the capital of the gambler after the 1st, 2nd $\cdots$, nth toss. The sequence $(a_1, a_2, \cdots, a_n)$ is called the path of the gambler's capital (or fortune) up to time n. Let $P(A \mid B)$ denote the conditional probability of an event A given that an event B has occured. The simple gambling scheme or the simple symmetric random walk $X_i, i = 1, 2, 3, \cdots$ is *Markovian*, which means that, for all $n \geq 1$, and any integer x, $0 \leq x \leq N$, and any $a_1, a_2, \cdots, a_n$,

$$P(X_{n+1} = x \mid X_1 = a_1, X_2 = a_2, \cdots, X_n = a_n)$$
$$= P(X_{n+1} = x \mid X_n = a_n).$$

In other words, the conditional probability distribution of X_{n+1}, given the path of the gambler's capital up to time n, depends entirely on the capital of the gambler *at* time n, and not on the path of her capital *up to* time n. That is, given the 'present', i.e., the value X_n at time n, the 'past', i.e., X_j, $j \leq n-1$ is not relevant for the probability distribution of the 'future', i.e., $X_{n+!}$.

In the next section we will discuss a gambling scheme more general than the simple symmetric random walk. This scheme allows the gambler to bet amounts more than one rupee, and at the same time allows her the freedom to decide on how much to bet (but not more than N) at any given time. Moreover, she can decide this amount (of how much to bet) based on the entire path of her capital up to that time. That is, she can be non-Markovian.

Since it is the careful investor, rather than an impulsive gambler, who takes into account the history of the market before readjusting her portfolio, we will change our language and speak in terms of sober and calculating investor rather than a compulsive gambler. It is important to note that an investor need not be an individual, but can be a company, small or large, a corporation, a government body, a charitable organization or trust or any other such institution.

3. An Investor's Martingale Walk: The Case of a Single Investor

We will now consider a simple generalization of the above model. At time 0 the investor invests an amount a, $0 < a < N$, with the hope of receiving a fixed higher amount b, $a < b \leq N$, at time 1 but is aware that she may loose in the process, and is willing to receive a smaller amount c, $0 \leq c < a$. The quantities a, b and c are chosen by the investor at the start. If she receives the amount b at time 1 then her gain g is $b - a$, while if she receives the amount c at time 1 then her loss l is $a - c$. Note that $b - c = b - a + a - c = g + l$. The

Martingale

The property that the average value of an investment portfolio at time $(n+1)$ over the random market fluctuations equals the value at time n is known as the martingale property. It is of great importance in financial mathematics.

probabilities $x = p(b)$, $y = p(c)$ that the investor receives the amount b or c at time 1 respectively are determined by the simultaneous equations:

$$x + y = 1, \quad (1)$$

$$bx + cy = a, \quad (2)$$

which gives

$$x = p(b) = \frac{a-c}{b-c} = \frac{l}{g+l}, \quad y = p(c) = \frac{b-a}{b-c} = \frac{g}{g+l}.$$

The requirement $bp(b)+cp(c) = a$ is known as *fair game* or *martingale* condition in probability theory. It is a consequence of *no arbitrage opportunities* requirement in financial mathematics. It says that on an average each investor neither loses or gains at each step. If this were not so and, say, the investor can expect to gain then a lot of investors will jump in and the gain will be lost. On the other hand if on an average you expect to lose then no investor will want to stay in the market. Thus (2) is a reasonable assumption.

We can rewrite this condition as $gp(b)-lp(c) = 0$. Since a is fixed, we may think of $p(b)$ also as the probability $P(g)$ of the investor making a gain of amount g while $p(c)$ may be interpreted as the probability $P(l)$ that the investor makes a loss of amount l, so that the fair game or martingale condition becomes

$$gP(g) - lP(l) = 0,$$

which is interpreted to mean that on an average the investor will break even.

From the equations

$$P(g) = \frac{l}{g+l}, \; P(l) = \frac{g}{g+l},$$

we note at once that $l \leq g$ if and only if $P(g) \leq P(l)$, so that under the martingale condition (2) a 'large' gain is possible only with 'small' probability, while a 'small' gain is possible with 'high' probability, in which case the loss is 'big' should the investor be unlucky to loose. Thus, these equations may be viewed as equations of the statutory warning in fast forward and small print: 'mutual fund investments are subject to market risk, read the offer document carefully before investing'.

The term 'fair game' comes from classical betting considerations (see Doob [1], Feller [2]) where a gambler is supposed to be playing against a gambling house, and the role of the investor is replaced by that of the gambler. If a is the amount the gambler bets, and receives an amount $b > a$ if she wins and an amount $c < a$ if she loses, then the game is said to be favorable to the gambler if $bp(b)+cp(c) > a$, whereas it is said to be favorable to the gambling house if $bp(b) + cp(c) < a$, and, fair if the equality holds. Here, as before, $p(b), p(c)$ denote the probabilities of the gambler receiving the amount b and c respectively. We note that the 'fair game' condition also seems to be fair between any two investors, rich or poor, because the probabilities $P(g)$ and $P(l)$ of gain and loss depend entirely on g and l and not on the initial investment a. However, these conclusions of fairness between the market forces and the investor or between two investors are deceptive as we will see.

If the investor chooses to attempt to make full gain, i.e., chooses $b = N$, and is lucky enough to receive the amount N at time 1, then she is contented and does not invest any more. Similarly if she chooses $c = 0$ and is unlucky enough to loose, then at time 1 she has no capital to invest, so that she does not invest anymore. In case she receives an amount d at time 1 which can be either b or c, and if $0 < d < N$, then she invests the amount d hoping to obtain a fixed higher amount e, $d < e \leq N$ at time 2, but is willing to receive a fixed smaller amount f, $0 \leq f < a$, should she loose. The probabilities $p(e), p(f)$, whose sum is one, are again determined by the 'fair game' condition $fp(f)+ep(e) = d$. The process continues. Some care is required to describe the situation at time n. Let x_i denote the amount the investor receives at time i, or already has this amount at time i, (which is the case if $x_{i-1} = 0$ or N). If at time n the investor has amount x_n and if $x_n = 0$ or N, then she does not invest anymore. Otherwise $0 < x_n < N$, in which case she invests this amount again choosing new quantities, say α and β, $0 \leq \alpha < x_n$, $x_n < \beta \leq N$, which she is willing to receive at time $n + 1$.

These quantities can depend on $x_0 = a, x_1, x_2, \cdots, x_n$, since the investor chooses α and β keeping in mind the history of the market and the world up to time n. The probabilities $p(\alpha)$ and $p(\beta)$ with which these values are realized satisfy (1) and (2), i.e.,

$$p(\alpha) + p(\beta) = 1, \tag{3}$$

$$\alpha p(\alpha) + \beta p(\beta) = x_n \quad \text{(fair game condition).} \tag{4}$$

Thus,

$$p(\alpha) = \frac{\beta - x_n}{\beta - \alpha}, \quad p(\beta) = \frac{x_n - \alpha}{\beta - \alpha}. \tag{5}$$

Let $p(x_1, x_2, \cdots, x_n)$ denote the probability that the investor receives an amount x_1 at time 1, x_2 at time 2, and in general, an amount x_n at time n. Let $p(x_{n+1} \mid x_1, x_2, \cdots, x_n)$ denote the conditional probability that the investor receives an amount x_{n+1} at time $n+1$ given that she has received an amount x_1 at time 1, x_2 at time 2, and in general an amount x_n at time n.

Recall that for any two events A and B, the probability of A and B happening together, $P(A \cap B)$, and the conditional probability of A given B, $P(A \mid B)$, satisfy, $P(A \cap B) = P(A \mid B) \cdot P(B)$. Thus,

$$p(x_1, x_2, \cdots, x_n)$$
$$= p(x_n \mid x_1, x_2, \cdots, x_{n-1}) p(x_1, x_2, \cdots, x_{n-1}).$$

On iteration we have

$$p(x_1, x_2, \cdots, x_n) = \Pi_{i=1}^{n} p(x_i \mid x_1, x_2, \cdots, x_{i-1}), \qquad (6)$$

where $p(x_1 \mid x_0)$ is interpreted to mean $p(x_1 \mid a) = p(x_1)$ since $p(a) = 1$. Now, from (5), we see that $p(x_i \mid x_1, x_2, \cdots, x_{i-1})$ is a ratio of two positive integers between 1 and N, so that

$$p(x_i \mid x_1, x_2, \cdots, x_{i-1}) \geq \frac{1}{N}.$$

Hence from (6),

$$p(x_1, x_2, \cdots, x_n) \geq \left(\frac{1}{N}\right)^n.$$

Let A_n denote the set of paths $(x_1, x_2, \cdots, x_n)$ with $x_n = 0$ or $x_n = N$, and let B_n = remaining set of paths, namely those paths $(x_1, x_2, \cdots, x_n)$ of length n for which $0 < x_n < N$. Note that for a path $(x_1, x_2, \cdots, x_n)$ in B_n, $0 < x_i < N$, for all $i, 1 \leq i \leq n$.

There is a non-increasing path $a \geq x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq \cdots$ wherein there are strict inequalities until an x_i is 0, after which they are all equalities. Moreover, the first i for which $x_i = 0$ is at most equal to $a < N$. There is a non-decreasing path $a \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots$ wherein there are strict inequalities until an x_i is N, after which they are all equalities. Moreover, the first i for which $x_i = N$ is at most equal to $N - a < N$. So the probability of the set of paths $(x_1, x_2, \cdots, x_N)$ with $x_N = 0$ or N is $> 2\left(\frac{1}{N}\right)^N$, i.e.,

$$p(A_N) > 2\left(\frac{1}{N}\right)^N,$$

whence

$$p(B_N) < 1 - 2\left(\frac{1}{N}\right)^N .$$

Consider now a path $(x_1, x_2, \cdots, x_N)$, of length N, with $0 < x_N < N$. Then the probability of the set of paths starting at x_N at time N and not hitting 0 or N during time points $(N+1, N+2, \cdots, 2N)$ is again $< 1-2\left(\frac{1}{N}\right)^N$. This implies that

$$p(B_{2N} \mid B_N) < 1 - 2\left(\frac{1}{N}\right)^N .$$

We see therefore that

$$p(B_{2N}) = p(B_{2N} \mid B_N) \cdot p(B_N) < \left(1 - 2\left(\frac{1}{N}\right)^N\right)^2 .$$

In general we have,

$$p(B_{kN}) < \left(1 - 2\left(\frac{1}{N}\right)^N\right)^k , \; k = 1, 2, \cdots .$$

Hence

$$p(B_{kN}) \to 0 \quad \text{as } n \to \infty.$$

In addition, $p(B_n)$ is non-increasing in n, hence $p(B_n) \to 0$ as $n \to \infty$.

This implies that

$$p\left(\bigcap_{n\geq 1} B_n\right) = \lim_n p(B_n) = 0.$$

But

$$B \equiv \bigcap_{n\geq 1} B_N$$

is the event that the gambling does not terminate in finite time. Thus, we have proved the first part of:

Theorem 3.1 (i) *The probability of the set of an investor's paths which hit* 0 *or* N *at some finite time is one, so that with probability one the investor will, in finite amount of time, either go bankrupt or reach her goal* N.

(ii) *The probability that the investor reaches* N *in finite time is* $\frac{a}{N}$, *and the probability that she reaches* 0 *in finite time is* $1 - \frac{a}{N}$.

Next we prove (ii). Let us be more mathematical. Write X_n for the amount the investor receives or has at time n. Given $X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1}$, we know that X_n assumes at most two values and the probabilities with which these values are assumed satisfy the martingale condition (4) with n replaced by $n-1$. We now write this as a conditional expectation:

$$E(X_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1}) = x_{n-1}.$$

We abbreviate $E(X_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1})$ as $E(X_n \mid x_1, x_2, \cdots, x_{n-1})$. Writing $E(X)$ for the expected value of a random variable X, we see that

$$\begin{aligned} E(X_n) &= \sum_C E(X_n \mid x_1, x_2, \cdots, x_{n-1}) p(x_1, x_2, \cdots, x_{n-1}) \\ &= \sum_C x_{n-1} p(x_1, x_2, \cdots, x_{n-1}) \\ &= \sum_{x_{n-1} \in D} x_{n-1} p(X_{n-1} = x_{n-1}) = E(X_{n-1}) \end{aligned}$$

where C is the set of investor paths up to time $n-1$, while D is the range of random variable X_{n-1}. We thus have

$$E(X_n) = E(X_{n-1}) = \cdots = E(X_1) = a. \qquad (7)$$

We have seen in (i) that $\lim_{n\to\infty} X_n = X_\infty$ exists with probability one and X_∞ assumes only two values 0 and N. Also

$$\mid EX_n - EX_\infty \mid \leq E \mid X_n - X_\infty \mid \leq 2NP(B_n).$$

By the first part of the theorem, $P(B_n) \to 0$ as $n \to \infty$, and since, by (7), $E(X_n) = a$ for all n, we see that

$$E(X_\infty) = a.$$

If s and t denote the probabilities with which X_∞ assumes values 0 and N respectively, then we have

$$E(X_\infty) = 0 \cdot s + t \cdot N = a,$$

so that

$$t = p(X_\infty = N) = \frac{a}{N}, s = p(X_\infty = 0) = 1 - \frac{a}{N}.$$

This proves the second part and completes the proof of the theorem.

A reader familiar with advanced probability will note that the above theorem follows from Doob's Martingale convergence theorem, since the process $X_n, n = 1, 2, \cdots$ is a uniformly bounded martingale (see [1], [3]).

Thus our model of an investor's walk, though hypothetical, and does not take into consideration the complexity of the market, does confirm the practical advice that honest investment advisers give to the middle class investors in India:

"If you do not have much money and your livelihood depends on what you earn on your savings, or your margin of saving is small, then be careful, but if your livelihood is taken care of, and financially you are sufficiently secure, then it is a good idea to take chances with mutual funds, and possibly earn a larger return."

A special case of investor's walk, namely, symmetric random walk with absorbing barriers, is already discussed in Section 2 above, and its implication for a gambler is well discussed in the probability literature, even when the probabilities are not symmetric (see [2], Chapter

XIV). Our model above, though based on a martingale building block, is much more general than the simple symmetric random walk, and leads to the same conclusions in so far as the investor is concerned. Since the transition probabilities at each stage are completely arbitrary (but for the martingale requirement), they need not be Markovian; so the investor's walk we have discussed is not a random walk in the strict sense of the term. Hence we call it a *Martingale walk*.

A greedy investor, no matter how rich, will hit the bottom in the long run.

It is important to note that the probability $\frac{a}{N}$ of reaching N depends only on a and not on what strategy (bold or conservative) the investor adopts.

If the investor is greedy and not contented with receiving the amount N, and executes her martingale walk without an absorbing barrier at the upper end, then she will eventually hit zero with probability one, no matter how rich she is to begin with. To see this we note that $p(X_\infty = 0) = 1 - \frac{a}{N} \to 1$ as $N \to \infty$ no matter what the starting capital a.

4. Implications for the Population as a Whole

So far we have discussed the implications of our conclusions for an individual investor, locally, as they say in mathematics. Are there implications globally, or for society at large? Surely there are. A constant refrain of sensitive and observant individuals, whether in India or in an advanced western country, is that *"rich are getting richer and poor are getting poorer"* (see [4],[5],[6]). These individuals are not necessarily left-leaning, and these are not views expressed out of ideological considerations, but rather out of concern for what they see. Can one justify these views, especially when one sees an individual poor person doing well by sheer hard work, and a well-to-do person getting poor? It seems we can. For the refrain *'rich getting richer and poor getting poorer'* is only an imperfect articulation of an obvious statisti-

Law of Large Numbers

If you do not know the probability of 'heads' with a given coin then a reasonable thing to do is to toss it a large number *n* of times and look at the proportion of heads that you get as an estimate of *p*. The law of large numbers provides the justification for this by asserting that if the tosses are independent then this proportion converges to *p* with probability one as *n* goes to infinity. A more general version of this law is the basis for modern statistical inference.

cal consequence of Theorem 3.1 together with the *law of large numbers*. (It says that if an experiment results in one of two possible outcomes, say success S or failure F with probabilities p and $(1-p)$ respectively and if the experiment is repeated independently n times and p_n is the proportion of successes in these n repetitions, then p_n gets close to p with high probability that goes to one as n goes to infinity.) Indeed, we see that the collective invested wealth of the investors does not change much, but only gets redistributed more lopsidedly.

Imagine that the market has 200 investors, 100 of them well to do and the remaining hundred not so well to do. Assume that the maximum possible receivable amount is 10 units, i.e., $N = 10$. (A unit could be thousand, 10 thousand, 100 thousand, or a million or more.) Assume that each of the well to do investor invests 7 units, while those not so well to do invest 3 units each. (We will assume that the 'fair game' condition holds and the process $(X_n)_{n=1}^{\infty}$ of investor's earnings is a Martingale.)

Indeed, we see that the collective invested wealth of the investors does not change much, but gets redistributed more lopsidedly.

According to our theorem, the probability of a well to do investor reaching 10 units in the long run is $\frac{7}{10}$, and the probability that a well to do investor hits 0 is $\frac{3}{10}$. Assume that the investors act independently. Let W_1 and L_1 be the number of winners and losers among the well to do investors. Then by the law of large numbers W_1 is approximately 70, and they will reach 10, while L_1 is approximately 30, and they will hit 0. In contrast, if W_2 and L_2 are the number of winners and losers among the not so well to do investors, then $L_2 =$ approximately 70, they will hit 0, and, $W_2 =$ approximately 30, and they will hit 10. Thus originally there were no paupers among the investors, now there are nearly 100 paupers among them, nearly 70 of them are those who were not well to do to begin with. Nearly 30 among them are previously well to do investors. Also there are now nearly 100 very well to do and contented investors, nearly 30 among them were not so well to do in the beginning. We also note that the total amount investment of at time zero is $7\times 100+3\times 100 = 1000$ which remains nearly the same since according to the new distribution of earned or lost wealth the total wealth remains approximately $10 \times 100 + 0 \times 100 = 1000$.

One can refine these assertions further by using what is known as the central limit theorem (see [3]).

Remark. The question whether the gambler and the investor in this article should be male or female was decided by tossing a coin.

Suggested Reading

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