Lighthill and his Art of Mathematical Modelling

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Lighthill modeled the flow of blood through a narrow capillary involving the passage of individual red blood cells through it in a single file. The study involved the motion of relatively tightly fitting pellets of solid matter forced along distensible tubes. The equations governing the motion and suitable boundary conditions are formulated. The mathematical highlight of Lighthill's model is the use of the method of matched asymptotic expansion in six layers.

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1. Introduction

Sir James Lighthill had the ability to translate a number of natural phenomena into mathematical systems, the solutions of which explained the behaviour of the physical system. In his work on aeroacoustics, the propagation of shock waves, the swimming of fishes, the flying of a humming bee and biofluiddynamics, he has described various problems that arise in these fields in terms of systems of differential equations. What is even more interesting is that the solutions of these systems are not the pedantic, usual ones, but exciting mathematical techniques specially crafted and suited to each problem.

In a paper entitled "Pressure-forcing of tightly fitting pellets along fluid filled elastic tubes", Lighthill modeled the flow of red blood cells being squeezed through a narrow capillary in single file lubricated by plasma. The human red blood cell in the unstressed state is a biconcave disc of diameter about 8 μm, but under stress it can be deformed. The narrowest capillaries are modeled as elastic tubes with internal diameter between 5

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and 10 μm with Poisson's ratio close to 0.5 with the local distension varying linearly with the local excess pressure. The red cells typically occupy 45 % of the blood volume, most of the remainder being occupied by the blood plasma, which can be treated as a Newtonian fluid.

Hydrodynamic lubrication theory is used to study the flow of pellets. This is observed to break down when the film thickness demanded by the dynamics is too small leading to "seize up". The slow forcing of fluid through the tube wall in regions of elevated pressure may lead to seizing-up process.

2. Mathematical Model

An exceedingly simplified dynamical model is proposed. The loss of fluid through the vessel walls is neglected, and the whole motion is taken to be symmetrical about the axis. However, a number of essential features have been retained in the model.

1. A pressure distribution with a local peak must exist while the pellet moves along the tube.
2. The thickness of the gap must be such that the viscous forces on the fluid within it balance the gradient of the pressure distribution.
3. The pressure forces and the viscous forces, which constitute the external forces must be in equilibrium.

The equations governing the motion are the laws of lubrication theory. The maximium diameter of the pellet is supposed to be greater or less than that of the tube by only a small fraction. The lubrication theory is a good approximation and it is only the parts of the surface at a distance from the axis nearly equal to the tube radius whose shape significantly influences the lubrication problem. These limited parts will be characterised geometrically to a good enough approximation by the curvature κ of the pellet's meridian section at the point of maximum diameter.

The Cross-Section: The meridian section at a certain reference pressure p_0 is described by

$$r = r_0 - \frac{1}{2}\kappa x^2, \tag{1}$$

where x is measured axially downstream from the point where the pellet cross-section has its maximum radius r_0.

The Elastic Properties of the Tube: The inner diameter of the tube is taken to increase linearly with the local pressure, but not to depend on the pressure at any other point. This is a reasonable assumption for tube materials with Poisson's ratio close to 0.5. A linear relation is assumed between the internal radius of the tube wall r and the local pressure p as

$$r = r_0 + \alpha(p - p_0), \tag{2}$$

where p_0 is the reference pressure. Pressure distributions with $p > p_0$ everywhere correspond to pellets whose maximum diameter is less than the minimum internal diameter of the tube, while distributions including regions where $p < p_0$ correspond to tubes whose diameter is in places less than the pellet's maximum diameter.

The expression for the cross-section of the pellet (1) is then modified to account for the elastic properties of the pellet. This gives

$$r = r_0 - \frac{1}{2}\kappa x^2 - \beta(p - p_0) \tag{3}$$

The thickness h of the lubricating film between the pellet and the inner tube wall is given as the difference between the two radii (2) and (3) as

$$h = (\alpha + \beta)(p - p_0) + \frac{1}{2}\kappa x^2. \qquad (4)$$

Since this thickness must be positive, the point $x = 0$ of maximum pellet diameter $p - p_0$ must be positive, although other regions may exist where $p < p_0$.

The pellet is being forced along the tube by fluid pressure difference at a velocity U. We choose a frame of reference which moves with the pellet at velocity U. In this frame of reference there is a steady axial flow which takes the value zero at the pellet and the value $(-U)$ at the tube wall.

The equations governing the flow are that of hydrodynamic lubrication theory. The Reynolds number based on film thickness is assumed small enough so that the inertial terms can be neglected in the equations of motion. The pressure p is taken to depend on the axial coordinate x alone based on the approximation of boundary layer theory. The thickness h is taken to be much smaller than the reference radius r_0 of the pellet tube.

As in theories of boundary layer, a special coordinate y is introduced which denotes the distance across the layer and is measured from the surface of the pellet. The axial momentum equation is

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \mu \frac{\partial^2 u}{\partial y^2}. \qquad (5)$$

This equation is to be solved subject to boundary conditions

$$u = 0(y = 0), \quad u = -U(y = h) \qquad (6)$$

and to an equation of continuity

$$\int_0^h u \; \mathrm{d}y = -Q, \qquad (7)$$

where $2\pi r_0 Q$ represents a rate of leakback of fluid past the pellet.

The solution for u must be quadratic in y, since $\frac{\partial^2 u}{\partial y^2}$ is independent of y. This determines the solution uniquely as

$$u = U\left[2\frac{y}{h} - 3(\frac{y}{h})^2\right] - \frac{6Q}{h}\left[\frac{y}{h} - (\frac{y}{h})^2\right]. \qquad (8)$$

This gives

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{6\mu U}{h^2} + \frac{12\mu Q}{h^3}. \qquad (9)$$

This differential equation has solutions with definite limits $p(\infty)$ and $p(-\infty)$ as x becomes large.

This represents, in practice, values of the pressure such as would be found immediately ahead of and behind the region where the lubricating film is thin. Specifying the downstream pressure $p(\infty)$ would essentially determine the positive or negative clearance $(\alpha+\beta)(p(\infty) - p_0)$ of the pellet in the tube at that downstream pressure.

Q remains an unknown and can be determined using the fact that the axial forces on the pellet are in equilibrium, which permits the pellet to move with constant velocity U. The skin friction τ resisting the motion is given by

$$\tau = -\mu\left(\frac{\partial u}{\partial y}\right)|_{y=0} = -\frac{2\mu U}{h} + \frac{6\mu Q}{h^2}. \qquad (10)$$

Balancing the resisitance force on the pellet is the axial force due to the pressure difference acting over the cross sectional area. This gives

$$\pi r_0^2[p(-\infty) - p(\infty)] = 2\pi r_0 \int_{-\infty}^{\infty}(-\frac{2\mu U}{h} + \frac{6\mu Q}{h^2})\mathrm{d}x. \qquad (11)$$

The range of solutions that are physically relevant is restricted by the fact that both sides have to be positive. The layer must be dominated by a region with

$$2Q/U < h < 3Q/U. \qquad (12)$$

3. Reduction to Non-Dimensional Form

Reducing the equations to non-dimensional form has the advantage that one can consider approximations based on the values of the non-dimensional parameters. A good measure of the film thickness is $2Q/U$, that of the pressure is what is required to increase the film thickness by one standard amount and that for the axial distance is in terms of that required to increase the film thickness by half the standard amount.

$$H = \frac{h}{2Q/U}, \quad P = \frac{(\alpha+\beta)(p-p_0)}{2Q/U}, \quad X = (\frac{\kappa}{2Q/U})^{1/2}x. \tag{13}$$

The equations governing the flow then can be written as

$$\frac{\mathrm{d}P}{\mathrm{d}X} = L(H^{-3} - H^{-2}), \quad H = P + \frac{1}{2}X^2, \tag{14}$$

where

$$L = \frac{6\mu U(\alpha+\beta)}{(2Q/U)^{5/2}\kappa^{1/2}}. \tag{15}$$

The non-dimensional constant L plays an important role in determining the character of the solution. Another non-dimensional constant of importance is

$$C = 2Q/Ur_0, \tag{16}$$

which is small compared to 1. C also plays an important role, because it is a measure of the fractional reduction in flow due to leakback of fluid past the pellets. The balance equation for the pressure is given by

$$P(-\infty) - P(\infty) = LC\int_{-\infty}^{\infty}\left(H^{-2} - \frac{2}{3}H^{-1}\right)\mathrm{d}X. \tag{17}$$

The equations can in principle be solved uniquely if L is given along with $P(\infty)$. One can then determine $P(-\infty)$ and thus the pressure difference needed to push the pellet at velocity U.

The solutions are analysed both in the limit $L \to 0$ and in the limit $L \to \infty$.

4. Analysis for Small L

For small values of L, the differential equation can be solved by a regular perturbation analysis. If one sets $P(\infty) = \frac{1}{2}\alpha^2$, then the zero and first order solutions are given by

$$\text{Zero order}: P = \frac{1}{2}\alpha^2;$$

$$\text{First order}: P(-\infty) - P(\infty) = \frac{2\pi L}{\alpha^2}\left(\alpha^2 - \frac{3}{2}\right).$$

We also obtain from the skin friction balance equation

$$\int_{-\infty}^{\infty} (H^{-2} - \frac{2}{3}H^{-1})\mathrm{d}X = \frac{4\pi}{3\alpha^3}\left(\frac{3}{2} - \alpha^2\right). \qquad (18)$$

To this order there is no band of solutions for which both quantities are positive. One requires $P(\infty) > 3/4$ and the other $P(\infty) < 3/4$ for the pressure difference to be positive. So one has to go to the next order of approximation. To the next order of approximation, we get two conditions

$$P(\infty) > 0.75 - 0.1313L^2, \text{ and } P(\infty) < 0.75 - 0.1169L^2,$$

which gives an exceedingly narrow band of solutions. Since only small values of C are of interest, one gets the solution

$$P(-\infty) - P(\infty) = LCf(L), \quad f(L) \sim 0.0657L^2, \quad L \to 0$$

$$P(\infty) \sim 0.75 - 0.1313L^2, \quad L \to 0.$$

Numerical evaluation shows that this perturbation analysis for small L gives reasonable results.

5. Analysis for Large L

The analysis for large L is important especially because the cases when the clearance between the pellet and tube would be negative if both were at the downstream value of the pressure are found to be in this range.

What makes the study for large L interesting is the fact that it would be a singular perturbation analysis. In this case Lighthill uses the method of matched asymptotic solutions to obtain the solution. Even to obtain a first approximate solution for large L involves matching forms of asymptotic solution that are different in six different layers.

The idea of a boundary layer is attributed to Ludwig Prandtl, a fluid dynamicist from the Kaiser Wilhelm Laboratory in Göttingen, Germany. In studying the flow past a body for a slightly viscous fluid, he noted that the effects of viscosity, which made the fluid adhere to the solid surface, was confined to a small region neighbouring the surface. In the rest of the fluid, the effects of viscosity could be ignored. This thin layer adjoining the solid was called the 'boundary layer'. The fact that the order of the governing equations was reduced in the bulk of the flow, meant that all the boundary conditions could not be satisfied there. This required the introduction of a new variable in the boundary layer, which was of a different length scale and then one matched the 'inner' and 'outer' solutions in the transition zone in a technique called 'matched asymptotic expansion'.

A Simple Example of Matched Asymptotic Expansion: Consider the equation

$$\epsilon y'' + (1+\epsilon)y' + y = 0, \quad 0 < \epsilon < 1, \tag{19}$$

with boundary conditions $y(0) = 0, y(1) = 1$.

If we were to neglect terms of $\mathrm{O}(\epsilon)$ to the zeroth order, we get

$$y' + y = 0, \qquad (20)$$

which is a first order ordinary differential equation and so only one boundary condition can be satisfied. If we choose the boundary condition $y(0) = 0$, we get $y(t) = 0$, for all t, a solution which is of little interest. If we choose the boundary condition $y(1) = 1$, then we get the solution

$$y(t) = e^{1-t},$$

which evidently does not satisfy the condition at $t = 0$. We infer that there must be a boundary layer at $t = 0$. In a thin layer adjoining $t = 0$ there is a steep change in the value of y. We set $\tau = t/\epsilon$ and transform the independent variable to τ, then the equation in the variable τ is called the equation in the inner region:

$$y''(\tau) + y'(\tau) + \epsilon(y'(\tau) + y(\tau)) = 0, \qquad (21)$$

which to zeroth order has the solution

$$y(\tau) = B - Ce^{-\tau}.$$

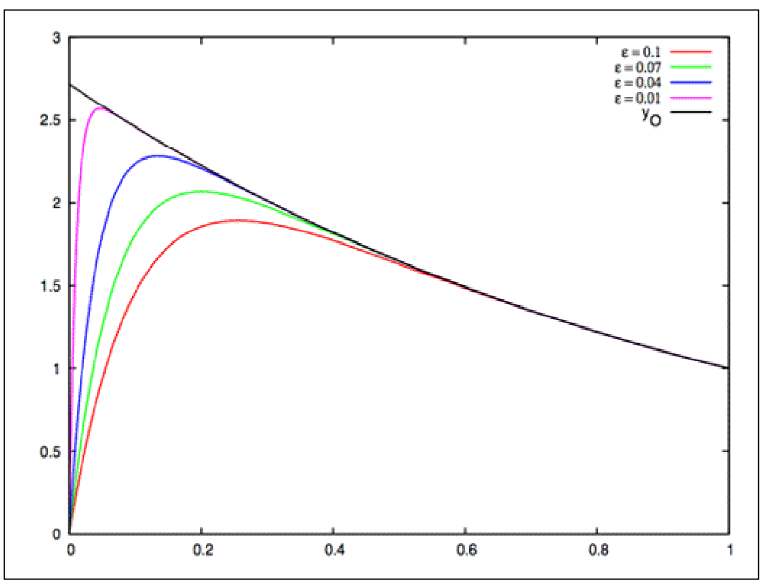

Figure 1. A plot of inner and outer solutions for various values of ε.

Since $y(0) = 0$, we have $B = C$. We now have two solutions: an outer solution $y_{\rm O}$ and an inner $y_{\rm I}$:

$$y_{\rm O} = e^{1-t}, \quad y_{\rm I} = C(1 - e^{-\tau}). \tag{22}$$

The inner and outer solutions have to match at the edge of the boundary layer as ϵ tends to zero. If we set $t = \sqrt{\epsilon}, \tau = 1/\sqrt{\epsilon}$ and let $\epsilon \to 0$, which is equivalent to $t \to 0, \tau \to \infty$, we get on matching $y_{\rm O}$, $y_{\rm I}$ that $C = e$. To obtain the final matched solution in the entire domain, add the inner and outer solutions and subtract their common value. This gives

$$y(t) = e^{(1-t)} - e^{(1-t/\epsilon)}. \tag{23}$$

This solution closely approximates the exact solution for small value of ϵ :

$$y(t) = \frac{e^{-t} - e^{-t/\epsilon}}{e^{-1} - e^{-1/\epsilon}}. \tag{24}$$

Six Layers for Matched Asymptotic Expansion: Even a first approximation for large L involves matching forms of asymptotic solution that are different in six separate layers. The six layers consist of three extended regions where the solution is very simple, namely a left-hand region extending to $X \to -\infty$, a central region including $X = 0$ and a right-hand region extending to $X \to \infty$. The first two are separated by a narrow region of simple boundary layer type, but the central and right region are separated by a double boundary layer.

Outermost and Central Regions: The two outermost regions are regions are constant pressure regions and can be approximated to a high order of approximation by

$$P = P(-\infty), \quad P = P(\infty),$$

in the left-hand region and right-hand regions, respectively. The central region is one of slowly varying film thickness and can be approximated near $X = 0$ up to order L^{-2} by

$$H = 1 + X/L.$$

Left-Hand Side Transition Layer: If the transition layer is centred on a point $X = -Y$ then the solution is given by

$$XY = P(-\infty) - \frac{1}{2}Y^2 - H - \int_H^\infty \frac{L(H-1)\mathrm{d}H}{YH^3 + L(H-1)}. \tag{25}$$

The thickness of this layer is of order $1/L$ and it terminates at the point $H = H_0$, where the denominator of the integrand $YH^2 + L(H-1)$ vanishes. The integral becomes logarithmically infinite as H approaches H_0. The boundary layer between the left-hand and central regions has thickness $1/L$ and merges with exponential rapidity into the central region. Y is determined as the value where H takes the intermediate value 1.

Right-Hand Side Transition Layer 1: There is a more complicated kind of boundary layer behaviour between the central and right-hand regions. The solution in the central region cannot be continued beyond $X/L = 4/27$, because $4/27$ is the maximum value that $H^{-2} - H^{-3}$ can take. We approximate around $X/L = 4/27$ using a quadratic approximation to $H^{-2} - H^{-3}$. This leads to the solution of a Ricatti equation for H:

$$\frac{\mathrm{d}H}{\mathrm{d}X} = X + L\left[\frac{16}{81}\left(H - \frac{3}{2}\right)^2 - \frac{4}{27}\right]. \tag{26}$$

The solution to this equation which fits on the left-hand side of $X/L = \frac{4}{27}$ into the value of H is given by

$$H = \frac{3}{2} + \left(\frac{81}{16L}\right)^{2/3}\chi\left[\left(\frac{16L}{81}\right)^{1/3}\left(\frac{4L}{27} - X\right)\right], \tag{27}$$

where $\chi(z)$ is the logarithmic derivative of the Airy function and

$$\chi(z) = Ai'(z)/Ai(z) \sim -\sqrt{z}, \qquad z \to +\infty.$$

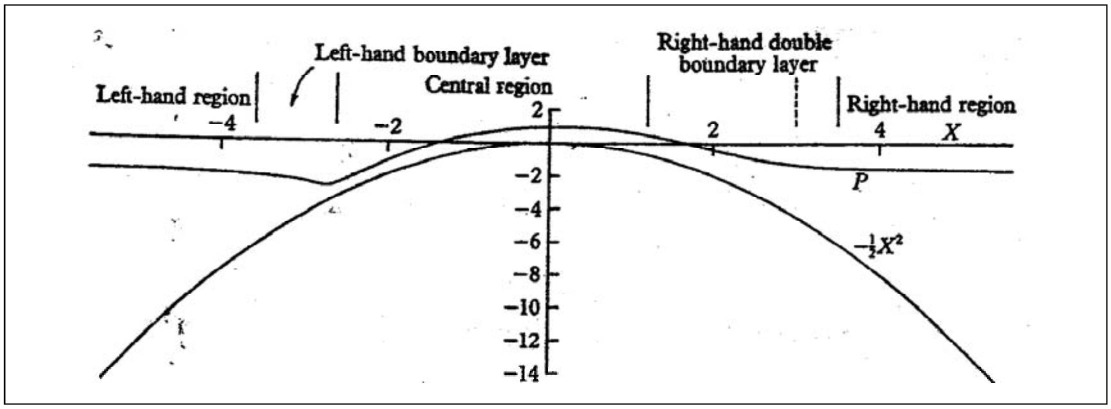

Figure 2. Pressure distribution P(X) and –½ X^2 for L = 8. The difference between the curves is the non-dimensional film thickness.

This solution is reasonable until X approaches

$$X_1 = \frac{4L}{27} + z_1 \left(\frac{81}{16L}\right)^{1/3}, \tag{28}$$

where z_1 is the smallest positive root of $Ai(-z) = 0$. A second boundary layer is required to fit this solution with the outermost constant right-hand region solution.

Right-Hand Side Transition Layer 2: This second right-hand side boundary layer can be worked out in the same way that the left-hand side boundary layer was done. The solution is approximated near $X = X_1$ and the one which merges with the right-hand solution gives

$$XX_1 = -P(\infty) + \frac{1}{2}X_1^2 + H - \int_H^{\infty} \frac{L(H-1)dH}{X_1 H^3 - L(H-1)}. \tag{29}$$

Matching solutions in a region where $H - \frac{3}{2}$ is relatively small, and $X - X_1$ is small for large enough L gives

$$XX_1 = -P(\infty) + \frac{1}{2}X_1^2 + \frac{3}{2} - G$$

$$+\frac{1}{\sqrt{z}}\left(\frac{81}{16L}\right)^{1/3} \arctan\left[\frac{H - \frac{3}{2}}{\sqrt{z_1}}\left(\frac{16L}{81}\right)^{2/3}\right]. \tag{30}$$

$$P(\infty) \doteq -\frac{1}{2}X_1^2 + \frac{3}{2} - G + \frac{\frac{1}{2}\pi}{\sqrt{z_1}}\left(\frac{81}{16L}\right)^{1/3} X_1,$$

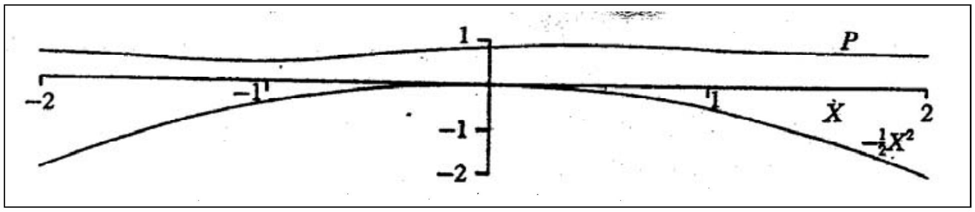

Figure 3. Pressure distribution P(X) and –½ X^2 for L = 1. The difference between the curves is the non-dimensional film thickness.

$$G = \int_{\frac{3}{2}}^{\infty} \frac{L(H-1)dH}{X_1 H^3 - L(H-1)}.$$

Numerical integration is performed in the 'stable' direction from left to right. Solutions for $L = 0.5$, 1, 2, 3, 4, 6 and 8 are obtained and plotted graphically.

6. Conclusion

Numerical calculations confirm that the pressure distribution and the film thickness are reasonably accurate and describe the physical phenomena correctly. The six layers predicted by the theory can be identified. There is fairly good agreement between the approximate theory for large L and computations for all L in the range $L > 3.2$. In his paper, Lighthill gives a detailed analyis of why the pressure distribution and the film thickness are as given by the theory and why the 'necking' at the rear of the pellet should not be confused with peristaltic propulsion of the pellet.

To model a physical phenomenon accurately one needs a person who has mastery of the laws of physics, who has a good command of mathematics and its application and finally, who is prepared to verify results using numerical analysis. Lighthill was a master of these three fields.

Suggested Reading

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[2] **J Kevorkian and J D Cole, *Multiple Scale and Singular Perturbation Methods*, Springer, 1996.**

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