# Algebraic Methods in Plane Geometry

## 3. The Use of Mappings

*Shailesh A Shirali*

Shailesh Shirali heads a Community Mathematics Center at Rishi Valley School (KFI). He has a deep interest in teaching and writing about mathematics at the high school/post school levels, with particular emphasis on problem solving and the historical aspects of the subject.

Part 1. The Use of Conic Sections, *Resonance*, Vol.13, No.10, pp.916–928, 2008.
Part 2. Cubic Curves, *Resonance*, Vol.13, No.11, pp.1027–1040.

**Keywords**
Felix Klein, Erlanger program, isometry, group, invariant, half turn, Napoleon's theorem, Morley's theorem, affine mapping.

**In this article we examine the role of mappings in elementary geometry. After making some comments about the Erlangen programme initiated by Felix Klein in 1872, wherein he proposed a way of studying geometries based on the underlying transformation groups, we see how theorems like Von Aubel's theorem and Napoleon's theorem can be proved in an elegant manner using similarity mappings, and how some construction problems may be solved using isometries. At the end we present a recent proof by Alain Connes of "Morley's Miracle", based on affine transformations.**

### 1. Felix Klein's Erlanger Programme

In 1872, when the great German mathematician Felix Klein was appointed professor at Erlangen, he prepared an inaugural address in which he set forth a unified view of all of geometry. This view over time became known as the *Erlanger program*, and it had a profound influence on the development of geometry. Klein's thesis was that *geometry is the study of those properties of a space that remain invariant under some given group of transformations.*

To illustrate what this means, consider the group of isometries in the plane. A mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be an *isometry* if it preserves distances: $d\,(f(\mathrm{P}), f(\mathrm{Q})) = d(\mathrm{P}, \mathrm{Q})$ for all pairs of points $\mathrm{P}, \mathrm{Q} \in \mathbb{R}^2$, where $d$ is the distance function. Reflection in a line is an example of such a mapping. It is clear that such a mapping must be one to one, and therefore invertible. In the next section we shall examine such mappings in more detail, but for now we note only that the set **Iso** of all such mappings forms a group in a natural manner, under functional composition; for, if $f, g, h \in \textsf{Iso}$, then $f \circ g \in \textsf{Iso}$; the inverse $f^{-1}$ of $f$ is in $\textsf{Iso}$; the identity map $\textsf{Id}$ is in $\textsf{Iso}$; and $f \circ (g \circ h) = (f \circ g) \circ h$, this relation being true for any three functions for which all the compositions are well defined. As all the group axioms are satisfied, $(\textsf{Iso}, \circ)$ is a group. It is called the *Euclidean group*.

Klein's Erlanger program had a profound influence on the development of geometry.

If we allow elements of this group to act on an arbitrary geometric object in $\mathbb{R}^2$, what features will the images share with one another? By definition, distances stay the same. This implies that notions like *collinearity* stay intact: if points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ lie in a straight line, so do their images $\mathrm{A}', \mathrm{B}', \mathrm{C}'$. Likewise for *perpendicularity*: if $l, m$ are lines perpendicular to each other ($l \perp m$), then their images $l', m'$ maintain that relation ($l' \perp m'$). More generally, *angles* stay unchanged: $\angle(l, m) = \angle(l', m')$; to see why, we only need to use the cosine rule. Such properties are *invariant* with respect to the Euclidean group, and under Klein's Erlanger programme, the study of all such properties may be said to constitute Euclidean geometry.

The notion of what 'constitutes Euclidean geometry' may seem imprecise, so we give an example of a property that does not belong here. Imagine that one introduces a notion of *balanced*: 'A triangle is balanced if its base is parallel to the $x$-axis'. By using a rotation (which is an isometry), we soon see that there are congruent pairs of triangle in which one triangle is balanced and the other is not. Hence, such a property is not invariant with respect to the Euclidean group, and so does not belong to Euclidean geometry.

Geometry is the study of those properties of a space that remain invariant under a given group of transformations.

If we enlarge the Euclidean group by including all *enlargements*, we get the group of similarity transformations, or the *group of mappings in the plane which preserve collinearity and leave angles unaltered.* In the language of complex numbers, such maps come in two forms, $z \mapsto az + b$ and $z \mapsto a\bar{z} + b$ ($a, b \in \mathbb{C}$, $a \neq 0$;

in the case of isometries, we have the further condition $|a| = 1$). The study of properties left invariant by this group constitutes similarity geometry. The various theorems of school geometry belong here.

The group of similarities is a subgroup of the *group of affine transformations*, under which the properties of collinearity and parallelism are preserved, but not distances and angles. (To see how this may come about, think of a *shear*.) And the similarity group in turn is a subgroup of the *group of projective transformations*, under which the properties of collinearity and incidence are preserved, but not parallelism. It may seem that all is lost by this stage, but this is not so: *cross ratios* are preserved, and this gives rise to many beautiful theorems of projective geometry; e.g., the theorems of Pascal and Brianchon.

## 2. Classes of Isometries

Obvious choices for isometries are the following: reflection in any line; rotation about any point, by any angle; displacement through any vector. It may be shown that compositions of these maps yield all possible isometries. In fact, the following stronger result is true: *Any isometry is a composition of no more than three reflections.* This is the 'three reflections theorem'.

The proof of this is based on the observation that an isometry is known if we know the images of any three non-collinear points. This in turn follows from elementary congruence theorems. For, let $f$ be an isometry which takes A to A$'$ and B to B$'$. Let C be any point not on $\overleftrightarrow{\mathrm{AB}}$. If its image under $f$ is C$'$, then $\triangle\mathrm{ABC} \cong \triangle\mathrm{A'B'C'}$. Hence, C$'$ lies on the circle with center A$'$ and radius $|\overline{\mathrm{AC}}|$, and also on the circle with center B$'$ and radius $|\overline{\mathrm{BC}}|$. This means that there are just two possible locations for C$'$. Once this location is fixed, the location of the image of any fourth point gets fixed because we know its distance from each of $\mathrm{A}', \mathrm{B}', \mathrm{C}'$; no more choice is left.

Any isometry is a composition of no more than three reflections.

| Symbol | Description |
|---|---|
| $\mathsf{D}_{\mathbf{v}}$ | Displacement through a vector $\mathbf{v}$ |
| $\mathsf{R}_{\mathrm{A},\theta}$ | Rotation about a point A, through an angle $\theta$ |
| $\mathsf{H}_{\mathrm{A}}$ | Half turn about a point A (hence, $\mathsf{H}_{\mathrm{A}} = \mathsf{R}_{\mathrm{A},180^\circ}$) |
| $\mathsf{M}_{\ell}$ | Reflection in a line $\ell$ |
| $\mathsf{E}_{(\mathrm{A},k)}$ | Enlargement about a point A, by a scale factor $k$ ($k \neq 0$) |

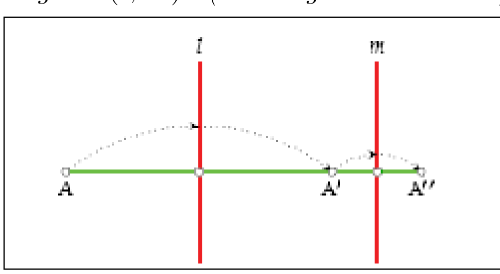
***Table 1. Some classes of mappings.***

Isometries are said to be *direct* if they preserve orientation (i.e, the sense of clockwise-anticlockwise); examples: rotations and displacements. If not, they are *indirect*; example: reflections. The three reflections theorem implies the following: *Every direct isometry is either a displacement or a rotation; every indirect isometry is either a reflection, or a reflection composed with a displacement, or a reflection composed with a rotation.*

In the language of complex numbers: isometries of the form $z \mapsto az + b$ ($|a| = 1$) are direct, and those of the form $z \mapsto a\overline{z} + b$ ($|a| = 1$) are indirect. For both forms, the rotational component of the isometry is captured by the argument of $a$.

The direct isometries form a subgroup of index 2 in the group ($\mathsf{\textbf{Iso}}, \circ$). With the notation as in *Table* 1, we find various relationships (Theorems 1–3) among these classes of mappings.

**Theorem 1** (Composition of reflections). *Let $l, m$ be distinct lines. Then $\mathsf{M}_m \circ \mathsf{M}_l$ is a displacement if $l \parallel m$, and a rotation if $l \nparallel m$.* ($\mathsf{M}_m \circ \mathsf{M}_l$ means: "Reflect in $l$, and then in $m$".) *In the former case, the displacement is through twice the directed distance from $l$ to $m$; in the latter case the angle of rotation is twice the directed angle $\angle(l, m)$. (See Figures 1 and 2.)*



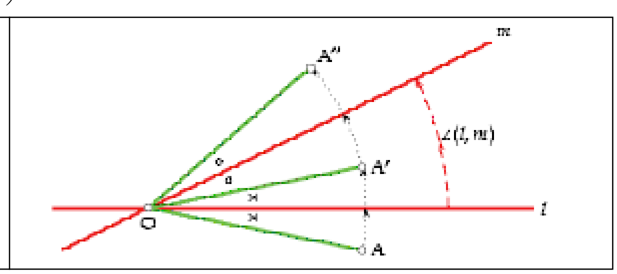




***Figure 1. (left) Reflections in two parallel mirrors.***
***Figure 2. (right) Reflections in two intersecting mirrors.***

The composition of an odd number of half turns is a half turn. The composition of an even number of half turns is a displacement.

**Corollary 1.1.** *Two reflections commute with each other if and only if their axes are perpendicular to each other, or they coincide.*

**Theorem 2** (Composition of rotations). *The composition of two rotations is either a rotation or a displacement: if* A, B *are arbitrary points, and* $\alpha, \beta$ *are arbitrary angles, then* $\mathsf{R}_{A,\alpha} \circ \mathsf{R}_{B,\beta}$ *is a rotation by* $\alpha + \beta$ *about some point* C*, unless* $\alpha + \beta$ *is a multiple of* $360°$*, in which case the composite map is a displacement.*

**Corollary 2.1.** *The composition of an odd number of half turns is a half turn. The composition of an even number of half turns is a displacement.*

**Theorem 3** (Characterization of similarity maps). *A map that preserves the relationship of equality of lengths is a similarity map. That is, if* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is such that for every collection of four points* P, Q, R, S *we have the following relation,*

$$d(\mathrm{P}, \mathrm{Q}) = d(\mathrm{R}, \mathrm{S}) \implies d(f(\mathrm{P}), f(\mathrm{Q})) = d(f(\mathrm{R}), f(\mathrm{S})),$$

*then* $f$ *is a similarity map.* (This means that for any three points P, Q, R, the triangle with vertices $f(\mathrm{P})$, $f(\mathrm{Q})$, $f(\mathrm{R})$ is similar to $\triangle \mathrm{PQR}$.)

Theorem 3 is far from obvious. We give only the proof of Theorem 2.

Given two distinct points A, B and two angles, $\alpha, \beta$, we must find what the composition $\mathsf{R}_{B,\beta} \circ \mathsf{R}_{A,\alpha}$ does. Let $l$ be the line $\overleftrightarrow{AB}$, let $m$ be the line through A such that $\angle(m, l) = \frac{1}{2}\alpha$, and let $n$ be the line through B such that $\angle(l, n) = \frac{1}{2}\beta$ (these are directed angles; see *Figure* 3); then $\angle(m, n) = \frac{1}{2}(\alpha + \beta)$. Theorem 1 tells us that $\mathsf{R}_{A,\alpha} = \mathsf{M}_l \circ \mathsf{M}_m$ and $\mathsf{R}_{B,\beta} = \mathsf{M}_n \circ \mathsf{M}_l$. Hence

$$\mathsf{R}_{B,\beta} \circ \mathsf{R}_{A,\alpha} = \mathsf{M}_n \circ \mathsf{M}_l \circ \mathsf{M}_l \circ \mathsf{M}_m = \mathsf{M}_n \circ \mathsf{M}_m.$$

If $\alpha + \beta \equiv 0 \pmod{360°}$ (this is the case, for example, if $\beta = -\alpha$), then $m \parallel n$, and the composite map is a

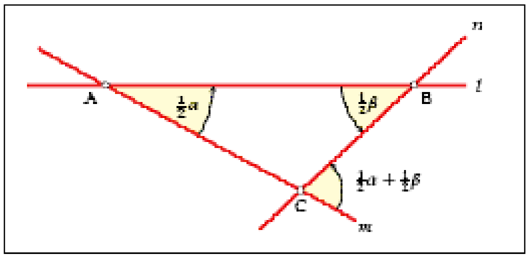



***Figure 3. Composition of two rotations.***

*displacement*. If not, then $m$ and $n$ meet at some point C, so that $\mathsf{M}_n \circ \mathsf{M}_m = \mathsf{R}_{\mathrm{C},\alpha+\beta}$. In this case, *the composite map is equivalent to the rotation* $\mathsf{R}_{\mathrm{C},\alpha+\beta}$.

Or, using complex numbers: let $f(z) = az + b$ and $g(z) = cz + d$ represent two rotations, with $|a| = 1 = |c|$, $a \neq 1$, $c \neq 1$; then $f \circ g(z) = acz + ad + b$. The fact that $|ac| = 1$, and the presence of $z$ rather than $\overline{z}$, tells us that this is a direct isometry. If $\arg a + \arg c \equiv 0 \pmod{2\pi}$ then $f \circ g$ is a displacement; else, it is a rotation. (*Remark.* This proof is more compact than the earlier one, but that proof also yields the location of the center of the composite map, and we shall see shortly how this knowledge can be of use.)

## 3. Groups Associated with Isometries

As noted earlier, the set **Iso** of all isometries in the plane forms the Euclidean group. This has various subgroups of interest. Consider the isometries that fix a particular point O. These are also the symmetries of any fixed circle with center O. These isometries form a subgroup of **Iso**; it is called the *orthogonal group* and is denoted by $\mathsf{O}(2)$. The components of this group are the rotations $\mathsf{R}_{\mathrm{O},\theta}$, with $\theta$ taking all possible values, and the reflections $\mathsf{M}_l$, for all possible lines $l$ through O. The rotations alone constitute a normal subgroup of $\mathsf{O}(2)$, called the *special orthogonal group* and denoted by $\mathsf{SO}(2)$; it is of index 2 in $\mathsf{O}(2)$. It is not hard to show that $\mathsf{O}(2)$ is isomorphic to the multiplicative group of all $2 \times 2$ matrices $A$ for which $AA^T = I_2$, where $I_2$ is the $2 \times 2$ identity matrix, while $\mathsf{SO}(2)$ is isomorphic to the multiplicative group of all $2 \times 2$ matrices $A$ for which $AA^T = I_2$, $\det A = 1$. The elements of $\mathsf{SO}(2)$ all have the form

The composition of two rotations is either a rotation or a displacement.

The group of all displacements is a normal subgroup of the group of isometries.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad (\theta \in \mathbb{R}),$$

while the elements of $\mathsf{O}(2)$ have the forms

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad (\theta \in \mathbb{R}).$$

The quotient group $\mathsf{O}(2)/\mathsf{SO}(2)$ is isomorphic to $(\{1, -1\}, \times)$.

Another important subgroup of **Iso** is the group $\mathcal{D}$ of all displacements in the plane; this is essentially the additive group of all vectors in the plane. In fact, $\mathcal{D}$ is a normal subgroup of **Iso**. To show this we must show that if $f$ is a displacement, and $g$ is an isometry, then $g^{-1} \circ f \circ g$ is a displacement. Using complex numbers this becomes easy: let $f(z) = z + a$ and $g(z) = bz + c$, where $a, b, c \in \mathbb{C}$, $|b| = 1$; then:

$$g^{-1} \circ f \circ g(z) = g^{-1} \circ f(bz + c)$$
$$= g^{-1}(bz + c + a) = \frac{bz + c + a - c}{b} = z + \frac{a}{b},$$

which is a displacement. (Note that the proof shows that $\mathcal{D}$ is a normal subgroup of the similarity group.) The quotient group $\textbf{Iso}/\mathcal{D}$ is isomorphic to $\mathsf{O}(2)$.

## 4. The Use of Mappings in Constructions

The following problem yields nicely to the algebra of mappings: *Given the midpoints of the sides of an $n$-sided polygon, in proper order, construct the polygon.* In the case of a triangle ($n = 3$) the problem is easily solved using the midpoint theorem, but we eschew this approach and look at the problem algebraically, as this enables us to find a unified approach that works for all $n \geq 3$.

Suppose then that $n = 3$; let the given points be P, Q, R. We must find points A, B, C such that P, Q, R are the midpoints of $\overline{\mathrm{BC}}$, $\overline{\mathrm{CA}}$, $\overline{\mathrm{AB}}$, respectively. Consider the

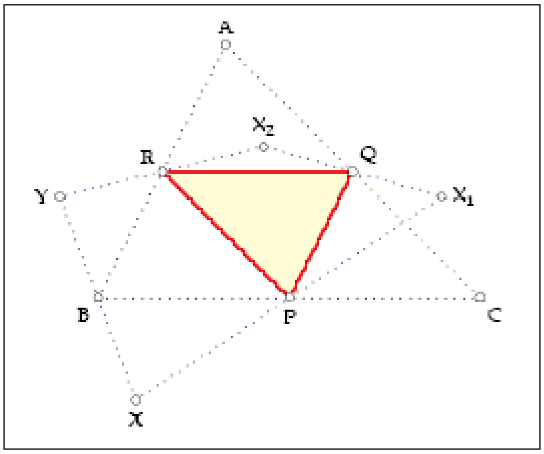



***Figure 4. Constructing △ABC from the midpoints P, Q, R of its sides.***

following map: $f = \mathsf{H}_{\mathrm{R}} \circ \mathsf{H}_{\mathrm{Q}} \circ \mathsf{H}_{\mathrm{P}}$. As $f$ is a composition of an odd number of half turns, it is itself a half turn. Now observe the effect of $f$ on B: the maps $\mathsf{H}_{\mathrm{P}}, \mathsf{H}_{\mathrm{Q}}, \mathsf{H}_{\mathrm{R}}$ applied in turn take B through the orbit $\mathrm{B} \mapsto \mathrm{C} \mapsto \mathrm{A} \mapsto \mathrm{B}$, implying that $f(\mathrm{B}) = \mathrm{B}$. Hence, B is the center of $f$. To locate B, it suffices to apply $f$ to any 'test point' X: if $f$ maps X to Y, then B is the midpoint of $\overline{\mathrm{XY}}$. Having found B, we now find C and A using the relations $\mathsf{H}_{\mathrm{P}}(\mathrm{B}) = \mathrm{C}$, $\mathsf{H}_{\mathrm{Q}}(\mathrm{C}) = \mathrm{A}$. *Figure* 4 illustrates the solution.

Note that: (a) a solution to the above problem exists for any three given points $\mathrm{P}, \mathrm{Q}, \mathrm{R}$ (if they are collinear, then $\mathrm{A}, \mathrm{B}, \mathrm{C}$ too will be collinear); and: (b) the same approach works for any odd value of $n$ ($n \geq 3$). For example, suppose we are given five points $\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}, \mathrm{T}$, and we want five points $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$ such that $\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}, \mathrm{T}$ are, respectively, the midpoints of $\overline{\mathrm{BC}}$, $\overline{\mathrm{CD}}$, $\overline{\mathrm{DE}}$, $\overline{\mathrm{EA}}$, $\overline{\mathrm{AB}}$. Let $f$ be the half turn $\mathsf{H}_{\mathrm{T}} \circ \mathsf{H}_{\mathrm{S}} \circ \mathsf{H}_{\mathrm{R}} \circ \mathsf{H}_{\mathrm{Q}} \circ \mathsf{H}_{\mathrm{P}}$. The center of $f$ is B, and it is found in the same way as earlier, using any test point X. Generalizing, we see that if $n \geq 3$ is odd, and we are given $n$ points $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$, then there is just one set of $n$ points $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n$ such that $\mathrm{P}_i$ is the midpoint of the segment connecting $\mathrm{A}_i$ and $\mathrm{A}_{i+1}$, and these points may be found as described above.

If $n$ is even, however, the situation is quite different. Take the case $n = 4$; let the given points be $\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}$,

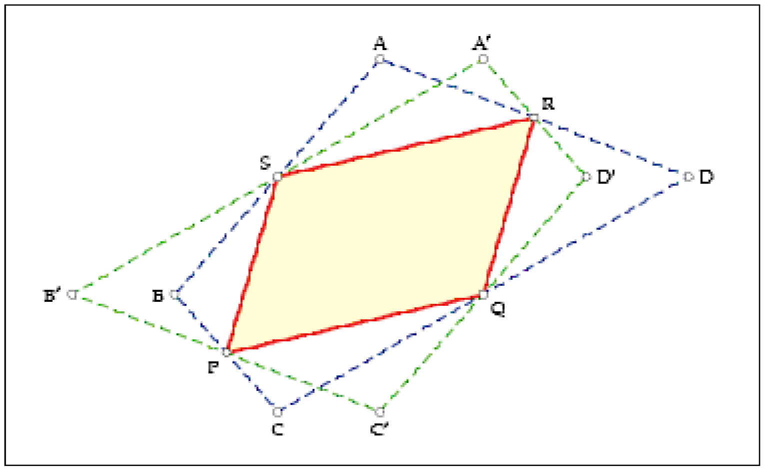



***Figure 5. Constructing quadrilateral ABCD from the midpoints P, Q, R, S of its sides: if PQRS is a parallelogram, there exist infinitely many solutions, otherwise there are none.***

and let $f = \mathsf{H}_{\rm S} \circ \mathsf{H}_{\rm R} \circ \mathsf{H}_{\rm Q} \circ \mathsf{H}_{\rm P}$. Then $f$ is a composition of an even number of half turns, and so is a *displacement*. If there exist four points A, B, C, D such that P, Q, R, S are, respectively, the midpoints of $\overline{\rm BC}$, $\overline{\rm CD}$, $\overline{\rm DA}$, $\overline{\rm AB}$, then, as earlier, B is a fixed point of $f$. *But a displacement with a fixed point is the identity map.* So for a solution to exist we must have the equality $\mathsf{H}_{\rm S} \circ \mathsf{H}_{\rm R} \circ \mathsf{H}_{\rm Q} \circ \mathsf{H}_{\rm P} = \mathsf{Id}$. This requires that $\overrightarrow{\rm PQ} = -\overrightarrow{\rm RS}$, i.e., PQRS must be a parallelogram. If this is the case, then for any point B, the quadrilateral ABCD whose vertices C, D, A are defined by $\rm C = \mathsf{H}_P(B)$, $\rm D = \mathsf{H}_Q(C)$, $\rm A = \mathsf{H}_R(D)$ is a valid solution; it has P, Q, R, S as the midpoints of its sides. So if one solution exists then there exist infinitely many solutions. See *Figure* 5.

The same situation maintains for $n = 6$; if the given points are P, Q, R, S, T, U, named in cyclic order, then a hexagon exists with these points as the midpoints of its sides if and only if the vector sum $\overrightarrow{\rm PQ} + \overrightarrow{\rm RS} + \overrightarrow{\rm TU}$ vanishes; and if this holds, then there are infinitely many solutions. The case for $n = 8, 10, 12, \ldots$ is similar.

## 5. The Use Of Mappings In Proving Theorems

Isometries and similarity maps yield elegant proofs for certain geometric results.

Isometries and similarity maps yield elegant proofs for certain geometric results. We consider two memorable examples below; the first one was posed as a problem in *Resonance* January 2008 (pp.35).

**Theorem 4** (Von Aubel). *The centers of squares drawn externally on the sides of a quadrilateral are the vertices of a quadrilateral whose diagonals are equal and perpendicular to each other.*

Given a quadrilateral ABCD, and squares ABEF, BCGH, CDIJ, DAKL drawn externally on its sides, let the centers of these squares be P, Q, R, S, respectively (see *Figure* 6). We must show that $\overline{\mathrm{SQ}} = \overline{\mathrm{PR}}$, $\overline{\mathrm{SQ}} \perp \overline{\mathrm{PR}}$.

For the proof, we use the quarter turns $f_{\mathrm{P}}, f_{\mathrm{Q}}, f_{\mathrm{R}}, f_{\mathrm{S}}$ defined by

$$f_{\mathrm{P}} := \mathsf{R}_{\mathrm{P},90^\circ},\ f_{\mathrm{Q}} := \mathsf{R}_{\mathrm{Q},90^\circ},\ f_{\mathrm{R}} := \mathsf{R}_{\mathrm{R},90^\circ},\ f_{\mathrm{S}} := \mathsf{R}_{\mathrm{S},90^\circ}.$$

Since the displacement $f_{\mathrm{Q}} \circ f_{\mathrm{R}} \circ f_{\mathrm{S}} \circ f_{\mathrm{P}}$ maps B back to itself (it is taken through the cycle B, A, D, C, B), it is the identity map. Hence, the half turns $f_{\mathrm{S}} \circ f_{\mathrm{P}}$ and $f_{\mathrm{Q}} \circ f_{\mathrm{R}}$ are identical. Let the center of each half turn be M (see *Figure* 6). Recalling how the center of the composition of two rotations is located (Theorem 2), we

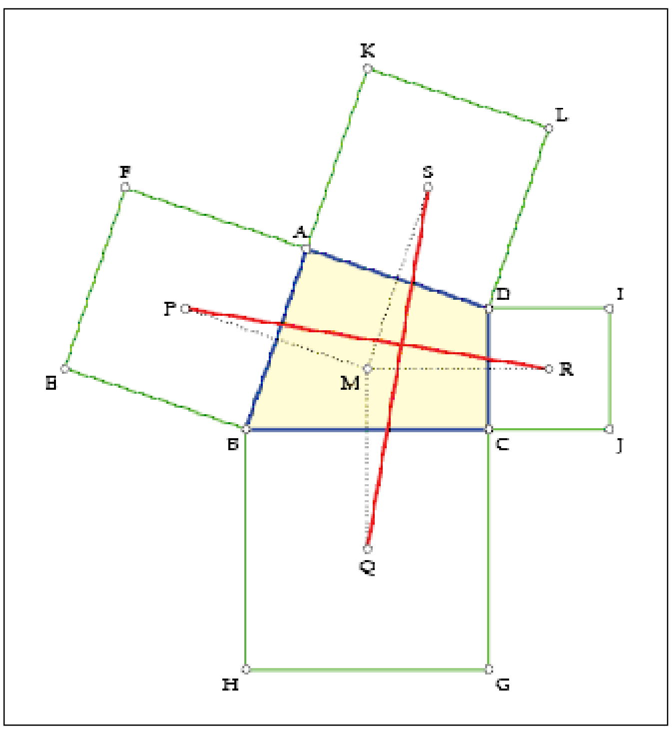



***Figure 6. Von Aubel's theorem: PR = QS, PR ⊥ QS.***

see that triangles MPS and MQR are isosceles, with a right angle at M. Hence, a quarter turn centered at M takes S to P, and Q to R. Consequently, it takes $\overline{\mathrm{SQ}}$ to $\overline{\mathrm{PR}}$, implying that $\overline{\mathrm{SQ}} = \overline{\mathrm{PR}}$, and $\overline{\mathrm{SQ}} \perp \overline{\mathrm{PR}}$.

**Theorem 5.** (Napoleon) *The centers of equilateral triangles drawn externally on the sides of a triangle are the vertices of an equilateral triangle.*

Given a triangle ABC, and equilateral triangles ABD, BCE, CAF drawn externally on its sides, let the centers of the triangles be P, Q, R, respectively; see *Figure* 7. We must show that triangle PQR is equilateral.

For $k = \frac{1}{\sqrt{3}}$, let $f, g$ be composite maps defined by

$$f := \mathsf{E}_{(\mathrm{B},k)} \circ \mathsf{R}_{\mathrm{B},30^\circ}, \qquad g := \mathsf{E}_{(\mathrm{C},k)} \circ \mathsf{R}_{\mathrm{C},-30^\circ}.$$

Each map is a composition of a rotation through 30° and an enlargement with scale factor $k$ (the two rotations are oppositely directed). Now observe that:

- $f$ takes D to P, and A to R; hence, $\overline{\mathrm{PR}} = k\,\overline{\mathrm{DA}}$, and $\angle(\overrightarrow{\mathrm{DA}}, \overrightarrow{\mathrm{PR}}) = 30^\circ$;
- $g$ takes D to P, and A to Q; hence, $\overline{\mathrm{PQ}} = k\,\overline{\mathrm{DA}}$, and $\angle(\overrightarrow{\mathrm{DA}}, \overrightarrow{\mathrm{PQ}}) = -30^\circ$.

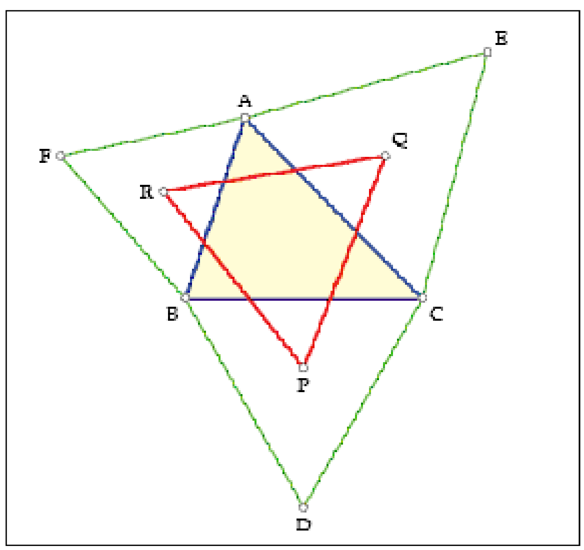



***Figure 7. Napoleon's theorem: $\Delta$ PQR is equilateral.***

Hence $\overline{\mathrm{PR}} = \overline{\mathrm{PQ}}$, $\angle(\overrightarrow{\mathrm{PR}}, \overrightarrow{\mathrm{PQ}}) = -60°$, and this implies that $\triangle$PQR is equilateral.

**Remark 1.** There are many features of interest in *Figure* 7; for example:

- Segments $\overline{\mathrm{AD}}$, $\overline{\mathrm{BE}}$, $\overline{\mathrm{CF}}$ have equal length, and they concur at a point T called the *Fermat-Toricelli point* of $\triangle$ABC; T has the property that if no angle of $\triangle$ABC exceeds 120°, then it is the point that minimizes the sum of the distances to the vertices of $\triangle$ABC.
- Segments $\overline{\mathrm{AP}}$, $\overline{\mathrm{BQ}}$, $\overline{\mathrm{CR}}$ concur as well, at $\mathrm{N}_1$, the *first Napoleon point* of $\triangle$ABC. If the equilateral triangles ABD, BCE and CAF are drawn so as to overlap with $\triangle$ABC (rather than lie outside it), then $\overline{\mathrm{AP}}$, $\overline{\mathrm{BQ}}$, $\overline{\mathrm{CR}}$ concur at $\mathrm{N}_2$, the *second Napoleon point* of $\triangle$ABC. See *Figure* 8.

If no angle of triangle ABC exceeds 120 degree, then T is the point that minimizes the sum of the distances to the vertices of the triangle.

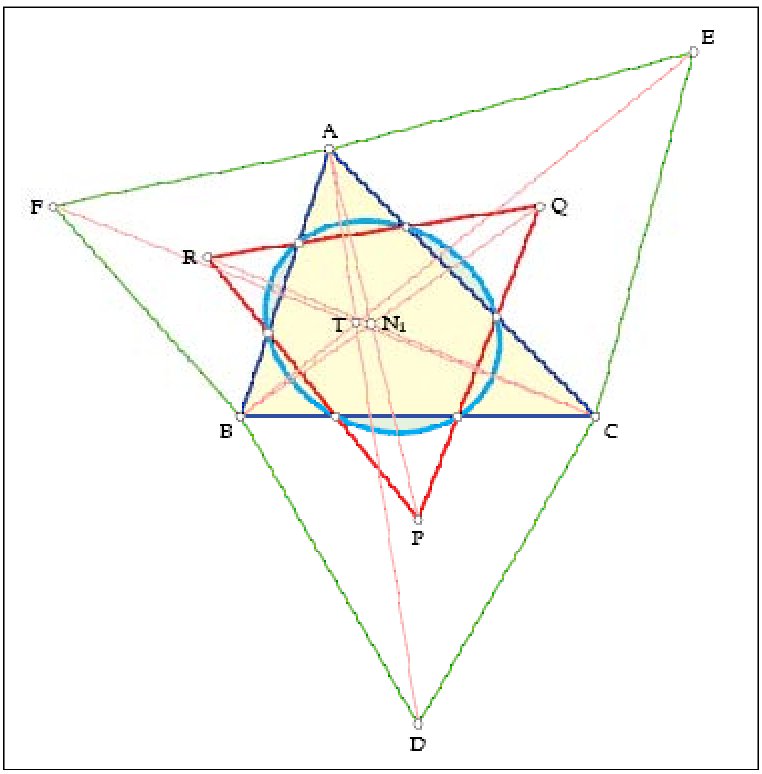



***Figure 8. Napoleon's theorem: $\Delta$PQR is equilateral.***

The well-known nine point circle theorem may also be proved in an elegant manner using transformations.

- Finally, the points of intersection of (BC, PQ), (BC, PR), (CA, QP), (CA, QR), (AB, RP), (AB, RQ) lie on a conic. We leave the proof to the reader. (Hint: Use the converse to Pascal's hexagon theorem.)

The well-known *nine point circle theorem* may also be proved in an elegant manner using transformations; but we leave this to the reader.

## 6. Morley's Miracle

We conclude with a recent proof of one of the great theorems of geometry: *Morley's theorem*. The proof is due to the 1982 Fields medalist Alain Connes.

**Thereom 6** (Frank Morley, 1896) *The three points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle.* See *Figure* 9.

The theorem was discovered by Frank Morley in 1896, as part of his work on algebraic curves tangent to a given number of lines; his proof was highly algebraic in nature. The nicest 'pure geometry' proofs are those due to M T Naraniengar (1909) and John Conway (1995); see [2].

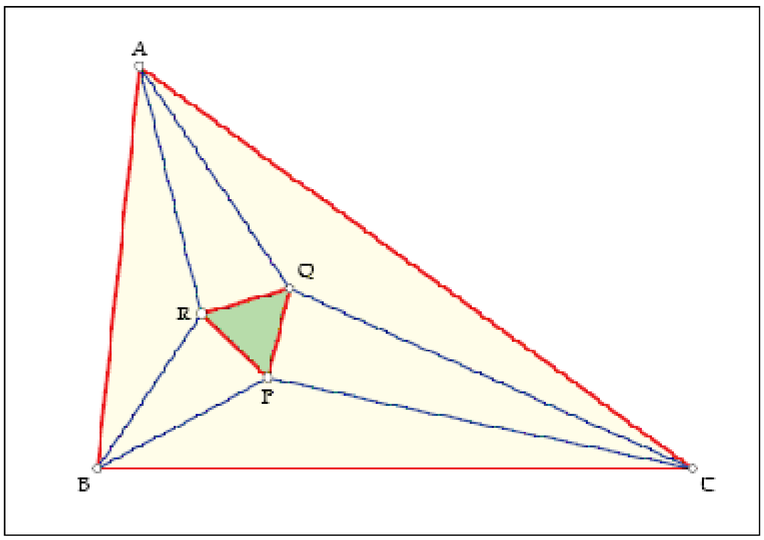



***Figure 9. Morley's miracle: $\Delta PQR$ is equilateral.***

Compact trigonometric proofs too may be found. But Connes's proof is of particular interest as it uses general affine mappings, and yields a result more general than Morley's theorem; see references [3],[4].

Connes's proof is of particular interest as it uses general affine mappings, and yields a result more general than Morley's theorem.

Let $\alpha = \frac{2}{3}\angle A$, $\beta = \frac{2}{3}\angle B$, $\gamma = \frac{2}{3}\angle C$, and define the rotations $f_a, f_b, f_c$ as follows: $f_a = \mathsf{R}_{\mathrm{A},\alpha}$, $f_b = \mathsf{R}_{\mathrm{B},\beta}$, $f_c = \mathsf{R}_{\mathrm{C},\gamma}$. Assume that the vertices of $\triangle$ABC are labeled in a positive ('counterclockwise') sense, as shown; then (Theorem 2),

- P is the fixed point of $\mathsf{R}_{\mathrm{B},\beta} \circ \mathsf{R}_{\mathrm{C},\gamma}$;
- Q is the fixed point of $\mathsf{R}_{\mathrm{C},\gamma} \circ \mathsf{R}_{\mathrm{A},\alpha}$;
- R is the fixed point of $\mathsf{R}_{\mathrm{A},\alpha} \circ \mathsf{R}_{\mathrm{B},\beta}$;
- $f_c^3 \circ f_b^3 \circ f_a^3 = \mathsf{Id}$; for, $f_a^3$ is equivalent to reflection in $\overleftrightarrow{\mathrm{AB}}$ followed by reflection in $\overleftrightarrow{\mathrm{BC}}$, and similarly for $f_b^3$ and $f_c^3$. A neat cancellation now takes place in the product $f_c^3 \circ f_b^3 \circ f_a^3$, giving the desired relation.

We shall show that Morley's theorem follows from these relations and a general result (Theorem 7) about affine functions defined on the complex plane $\mathbb{C}$.

An *affine function* $g : \mathbb{C} \to \mathbb{C}$ is a function of the type $g(x) = ax+b$, where $a, b \in \mathbb{C}$, $a \neq 0$. If $a = 1$, then $g$ is a translation; and if $|a| = 1$ but $a \neq 1$, then $g$ is a rotation. The set of all such functions under composition yields a non-abelian group $G$. If $g(x) = ax + b$ is in $G$, we define $\delta(g) = a$; then $\delta$ is a homomorphism from $G$ into the multiplicative group of non-zero complex numbers. For, if $g(x) = ax + b$ and $h(x) = cx + d$ are in $G$, then

$$h \circ g(x) = c(ax + b) + d = acx + (bc + d),$$

$$\therefore\ \delta(h \circ g) = ac = \delta(h)\, \delta(g).$$

The kernel of $\delta$ is the subgroup of pure translations in $G$. If $g(x) = ax + b$ is not a translation, i.e., $a \neq 1$, then $g$ has a unique fixed point given by

$$\mathrm{fix}(g) = \frac{b}{1-a}.$$

This yields a total of 18 different equilateral triangles associated with the trisectors of the angles of any triangle.

**Theorem 7** (Alain Connes) *Let $g_1, g_2, g_3 \in G$ such that $g_1 \circ g_2$, $g_2 \circ g_3$, $g_3 \circ g_1$, $g_1 \circ g_2 \circ g_3$ are not translations. Let $j = \delta(g_1 \circ g_2 \circ g_3)$, $p = \text{fix}(g_1 \circ g_2)$, $q = \text{fix}(g_2 \circ g_3)$, $r = \text{fix}(g_3 \circ g_1)$. Suppose that $g_1^3 \circ g_2^3 \circ g_3^3 = \mathsf{Id}$; then $j^3 = 1$ and $p^2 + q^2 + r^2 = pq + qr + rp$.*

*Proof.* Let $g_k(x) = a_k x + b_k$. After much calculation we get: $g_1^3 \circ g_2^3 \circ g_3^3(x) = a_1^3 a_2^3 a_3^3 x + (a_1^2 + a_1 + 1)\, b_1 + a_1^3 (a_2^2 + a_2 + 1)\, b_2 + (a_1 a_2)^3 (a_3^2 + a_3 + 1)\, b_3$. Hence the equality $g_1^3 \circ g_2^3 \circ g_3^3 = \mathsf{Id}$ is equivalent to $a_1^3 a_2^3 a_3^3 = 1$ and $b = 0$, where $b$ is the translational portion of $g_1^3 \circ g_2^3 \circ g_3^3$, given by:

$$b = \left(a_1^2 + a_1 + 1\right) b_1 + a_1^3 \left(a_2^2 + a_2 + 1\right) b_2 +$$

$$(a_1 a_2)^3 \left(a_3^2 + a_3 + 1\right) b_3.$$

Since $j = a_1 a_2 a_3$, the condition $a_1^3 a_2^3 a_3^3 = 1$ is the same as $j^3 = 1$. Showing that $p^2 + q^2 + r^2 = pq + qr + rp$ is tedious and unenlightening, and best left to a computer algebra system (see below for details).

The application to Morley's theorem is clear: the hypotheses of the theorem hold for the affine functions $f_a, f_b, f_c$, hence $p^2 + q^2 + r^2 = pq + qr + rp$. This well-known criterion for $p, q, r$ to be the vertices of an equilateral triangle shows that $\triangle$PQR is equilateral.

Connes's result implies the following two stronger results.

**Corollary 7.1.** *Let the vertices of $\triangle$ABC be labeled in a positive sense. For $\alpha', \beta', \gamma' \in \{0°, 120°, 240°\}$, let $\alpha = \frac{2}{3}\angle A + \alpha'$, $\beta = \frac{2}{3}\angle B + \beta'$, $\gamma = \frac{2}{3}\angle C + \gamma'$, $f_a = \mathsf{R}_{\mathrm{A},\alpha}$, $f_b = \mathsf{R}_{\mathrm{B},\beta}$, $f_c = \mathsf{R}_{\mathrm{C},\gamma}$. Let P, Q, R be, respectively, the fixed points of the rotations $\mathsf{R}_{\mathrm{B},\beta} \circ \mathsf{R}_{\mathrm{C},\gamma}$, $\mathsf{R}_{\mathrm{C},\gamma} \circ \mathsf{R}_{\mathrm{A},\alpha}$, $\mathsf{R}_{\mathrm{A},\alpha} \circ \mathsf{R}_{\mathrm{B},\beta}$. Then $\triangle$PQR is equilateral.*

This yields a total of 18 different equilateral triangles associated with the trisectors of the angles of any triangle. Corollary 7.2, which is Connes's result expressed in another way, shows that there are equilateral triangles associated with any pair of affine mappings, subject only to some mild restrictions.

**Corollary 7.2.** *Let complex numbers* $a_k, b_k$ $(k = 1, 2)$ *be such that* $a_1a_2 \neq 0$*. Let complex numbers* $a_3, b_3$ *be such that* $a_1^3a_2^3a_3^3 = 1$ *and*

$$\left(a_1^2 + a_1 + 1\right) b_1 + a_1^3 \left(a_2^2 + a_2 + 1\right) b_2 +$$

$$(a_1a_2)^3 \left(a_3^2 + a_3 + 1\right) b_3 = 0.$$

*Suppose that* $a_1a_2 \neq 1$, $a_2a_3 \neq 1$, $a_3a_1 \neq 1$, $a_1a_2a_3 \neq 1$. *Define* $p, q, r$ *as follows:*

$$p = \frac{a_1b_2 + b_1}{1 - a_1a_2}, \quad q = \frac{a_2b_3 + b_2}{1 - a_2a_3}, \quad r = \frac{a_3b_1 + b_3}{1 - a_3a_1}.$$

*Then the points corresponding to* $p, q, r$ *are the vertices of an equilateral triangle.*

*Proof.* We simply offer a `Mathematica` verification:

```
ClearAll[a1,a2,b1,b2,w,a3,b3n,b3d,b3,p,q,r,z];
w = (-1 + I Sqrt[3])/2;
a3 = w/(a1 a2);
b3n = (a1^2 + a1 + 1)b1 + a1^3(a2^2 + a2 + 1)b2;
brd = a1^3 a2^3 (a3^2 + a3 + 1);
b3 = -brn/brd;
p = (a1 b2 + b1)/(1 - a1 a2);
q = (a2 b3 + b2)/(1 - a2 a3);
r = (a3 b1 + b3)/(1 - a3 a1);
z = p^2 + q^2 + r^2 - p q - q r - r p;
FullSimplify[z]
```

The answer is 0; it remains 0 if we change the third line to `a3 = w^2/(a1 a2)`. Since $z = 0$ (we shall not bother to argue with the result of a `Mathematica` simplification), it follows that the points corresponding to $p, q, r$ are indeed the vertices of an equilateral triangle.

**Remark 2.** Connes writes in [4], "The purpose of this short note is to give a conceptual proof of Morley's theorem as a group theoretic property of the action of the affine group on the line." He adds that the proof must make use of special Euclidean properties of the group of isometries, for Morley's theorem does not hold in non-Euclidean geometry.

**Remark 3.** Following Connes's approach, one may try to cast other results of plane geometry within a similar framework. We consider just one example. Let ABC be any triangle, its vertices labeled in a positive sense, and let $\alpha, \beta, \gamma$ be (respectively) the measures of its angles at $\mathrm{A}, \mathrm{B}, \mathrm{C}$. Define the rotations $f_a = \mathsf{R}_{\mathrm{A},\alpha}$, $f_b = \mathsf{R}_{\mathrm{B},\beta}$, $f_c = \mathsf{R}_{\mathrm{C},\gamma}$; then we have:

- $f_a^2 \circ f_b^2 \circ f_c^2 = \mathsf{Id}$;
- the rotations $f_a \circ f_b$, $f_b \circ f_c$, $f_c \circ f_a$ have the same fixed point (i.e., the same center), namely, the incenter I of $\triangle$ABC;
- $f_a \circ f_b \circ f_c$ is a half turn centered at the point where the incircle of $\triangle$ABC touches side CA.

## Suggested Reading

[1] I M Yaglom, *Geometric Transformations*, Volumes I, II, III, New Mathematical Library, Math. Assocn America, 1962.

[2] A Bogomolny, Morley's Miracle, from *Interactive Mathematics Miscellany and Puzzles*, http://www.cut-the-knot.org/content.shtml

[3] A Connes, *A View of Mathematics*, available as a pdf file from http://www.alainconnes.org/en/downloads.php

[4] A Connes, *A new proof of Morley's Theorem*, available as a pdf file from http://www.alainconnes.org/en/downloads.php

*Address for Correspondence*
Shailesh A Shirali
Rishi Valley School
Rishi Valley 517 352
Madanapalle, AP, India.
Email:
shailesh.shirali@gmail.com