Algebraic Methods in Plane Geometry

1. The Use of Conic Sections

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Written with affection and respect for Professor A R Rao of Ahmedabad, mathematician, teacher and a continuing source of inspiration to a vast number of students, on the occasion of his one hundredth birthday. May there be many more!

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'Riders' in geometry are always a pleasure to tackle, and this pleasure is doubled when one finds connections between plane geometry and algebra. This three-part article is about such connections. In Parts 1 and 2, we explore some connections between plane geometry and the algebra of conics and cubics; in Part 1 we give algebraic proofs of results such as Pascal's Theorem and the Butterfly Theorem, and in Part 2 we study some group theoretic and number theoretic aspects of cubic curves. In Part 3 we look at the role of mappings and transformation groups in plane geometry.

1. Parabola in a Triangle

We first recall two results from the geometry of the parabola. Let $\mathcal{P}$ denote a parabola with focus F and directrix ℓ. For any point $\mathrm{P} \in \mathcal{P}$, let t_{P} denote the tangent to $\mathcal{P}$ at P.

(i) The image of F under reflection in any of the tangents t_{P} lies on the directrix ℓ. (See *Figure* 1a).

Conversely, if the image of F under reflection in a line m lies on ℓ, then m is tangent to $\mathcal{P}$. (The collection of all such lines m envelopes the parabola in a visually very attractive way, as can be shown using paper folding.)

(ii) If $\mathrm{A}, \mathrm{B}, \mathrm{C}$ are three distinct points on $\mathcal{P}$, then the circumcircle of the triangle PQR whose sides lie on the tangents $t_{\mathrm{A}}, t_{\mathrm{B}}, t_{\mathrm{C}}$, respectively, passes through the focus F. (See *Figure* 1b).

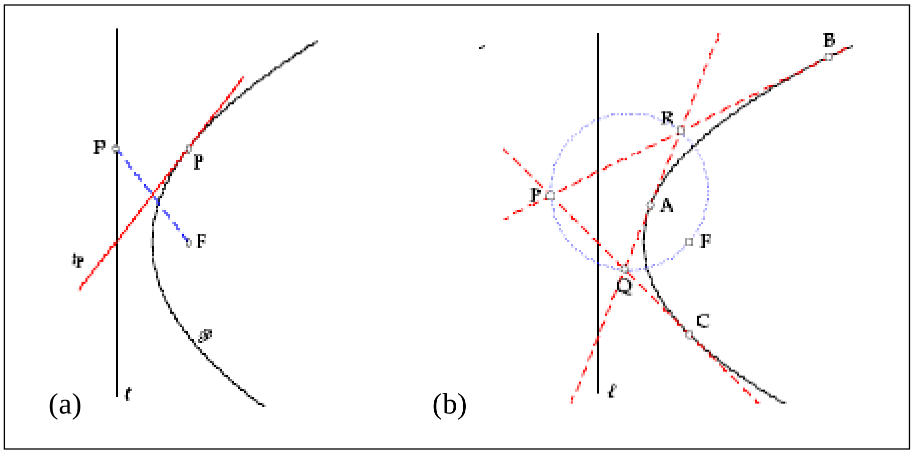

Figure 1.

Some readers may recognize that these two results come together in the *Wallace–Simson theorem*: "For any triangle the feet of the normals from a point on its circumcircle to the three sides of the triangle lie in a straight line".

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The result below, whose source is a problem from the problem solving magazine *Crux Mathematicorum*, brings these two results in a pretty way.

Theorem 1. *In triangle* ABC *let the feet of the altitudes from* A, B, C *be* D, E, F, *respectively. Let* $\overleftrightarrow{\mathrm{EF}}$ *cut* $\overleftrightarrow{\mathrm{AD}}$ *in* K, *let* L *be the midpoint of* $\overline{\mathrm{KD}}$, *and let the normal to* $\overline{\mathrm{AD}}$ *at* L *cut* $\overleftrightarrow{\mathrm{AC}}$ *in* Q, *and* $\overleftrightarrow{\mathrm{AB}}$ *in* R. *Then the points* A, R, D, Q *are concyclic.* (See *Figure* 2.)

Proof. Consider the parabola $\mathcal{P}$ with focus D, and directrix $\overleftrightarrow{\mathrm{EF}}$ (*Figure* 3). Observe that:

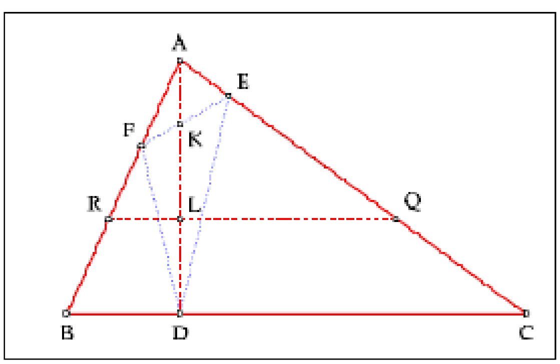

Figure 2.

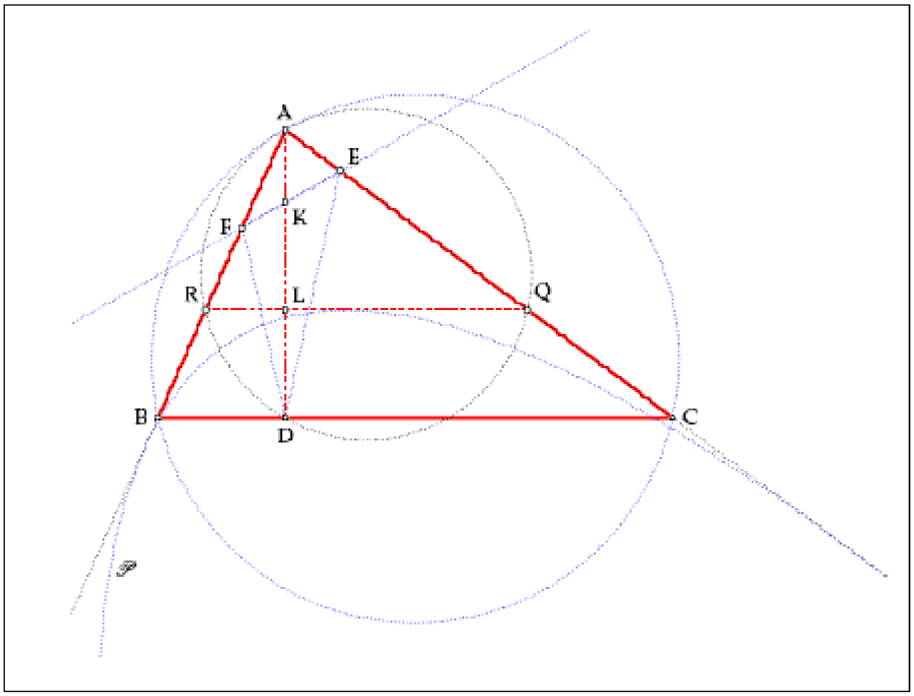

Figure 3.

It is a known property of a triangle formed by three tangents to a parabola that its circumcircle passes through the focus of the parabola.

- $\overleftrightarrow{\mathrm{QR}}$ *is tangent to* $\mathcal{P}$*.* This is because the image of D under reflection in $\overleftrightarrow{\mathrm{QR}}$ lies on the directrix $\overleftrightarrow{\mathrm{EF}}$. (The image is K.)
- $\overleftrightarrow{\mathrm{AB}}$ *is tangent to* $\mathcal{P}$*.* This is because the image of D under reflection in $\overleftrightarrow{\mathrm{AB}}$ lies on the directrix. In turn this is true because of the known property that the altitudes $\overline{\mathrm{AD}}$, $\overline{\mathrm{BE}}$, $\overline{\mathrm{CF}}$ bisect the angles of the orthic triangle $\triangle$ DEF. Hence, $\overline{\mathrm{FE}}, \overline{\mathrm{FD}}$ make equal angles with $\overleftrightarrow{\mathrm{AB}}$, implying that the image of D under reflection in $\overleftrightarrow{\mathrm{AB}}$ lies on $\overleftrightarrow{\mathrm{EF}}$, as claimed.
- $\overleftrightarrow{\mathrm{AC}}$ *is tangent to* $\mathcal{P}$*.* This is so because the image of D under reflection in $\overleftrightarrow{\mathrm{AC}}$ lies on the directrix (as above).
- *The circumcircle of* $\triangle$ ARQ *passes through* D*.* For, the tangents $\overleftrightarrow{\mathrm{RQ}}$, $\overleftrightarrow{\mathrm{AB}}$, $\overleftrightarrow{\mathrm{AC}}$ are the sides of $\triangle$ ARQ, and it is a known property of a triangle formed by three tangents to a parabola that its circumcircle passes through the focus of the parabola. □

Note that since $\overleftrightarrow{RQ}$ is parallel to $\overleftrightarrow{BC}$, the circumcircles of $\triangle$ ARQ and $\triangle$ ABC will be tangent to each other at A, as *Figure* 3 shows.

2. Families of Curves

We now state an extremely useful result concerning *families of curves with a given degree*. The entire discussion is with reference to a fixed rectangular coordinate system on a given plane.

Consider the family $\mathcal{L}$ of all possible straight lines. The general equation of a straight line is $ax+by+c=0$ (with a, b not both zero). There are three coefficients in this equation, but if we multiply all of them by a non-zero constant we get the same line. Hence, $\mathcal{L}$ has 'two degrees of freedom' (to borrow an expression from physics); or phrased otherwise, $\mathcal{L}$ is a two parameter family, and two distinct points are needed to fix a straight line. If we fix one point and allow the other one to vary, then we get the one parameter family of all lines passing through a point, also called a 'pencil of lines'. The term 'pencil' may be understood from *Figure* 4a.

Next, consider the family $\mathcal{F}$ of all possible second degree curves, i.e., all possible curves of the type $p(x, y) = 0$, where

$$p(x, y) \;=\; ax^2 + by^2 + cxy + dx + ey + f.$$

Observe that p has $3 + 2 + 1 = 6$ parameters. If we multiply p by a non-zero constant, the curve remains

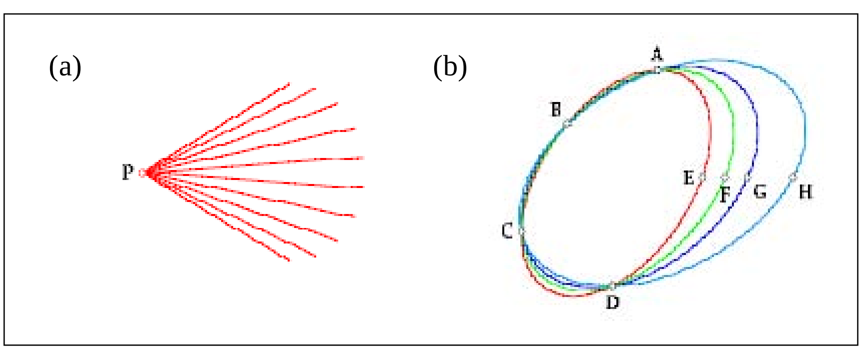

Figure 4.

Five points in general position are needed to fix a second degree curve.

the same, so $\mathcal{F}$ is a five-parameter family. The geometrical implication of this is that *five points in general position are needed to fix a second degree curve*. (In contrast, the family of circles has three degrees of freedom, and three points in general position are needed to fix a circle. In both cases, 'general position' means that no three points lie in a straight line.) If we fix four points and allow the fifth one to vary, we get a one parameter family (a 'pencil of conics'), as the sketch in *Figure* 4b shows.

A simple but extremely useful corollary to the above observation is the following: *If* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are two second degree curves passing through four given points* A, B, C, D, *with equations* $p_1(x, y) = 0$ *and* $p_2(x, y) = 0$, *respectively, then the equation of any other second degree curve* $\mathcal{C}_3$ *passing through* A, B, C, D *may be written in the form* $\alpha_1 p_1(x, y) + \alpha_2 p_2(x, y) = 0$, *where* α_1, α_2 *are real constants, not both zero.* Similar statements may be made about the pencil of lines passing through a fixed point, or about the family of circles passing through two given points. (This principle extends to cubics as well, but we shall study this only in Part 2.)

3. Pascal's Hexagram Theorem

We now prove a famous and important theorem about the conic sections, due to Blaise Pascal (1623–1662).

Theorem 2 (Pascal)**.** *The opposite sides of a hexagon inscribed in a conic intersect in three collinear points.* (See *Figure* 5.)

That is, if six distinct points A, B, C, D, E, F lie on a conic $\mathcal{C}$, then the points of intersection $\mathrm{P} = \overleftrightarrow{\mathrm{AB}} \cap \overleftrightarrow{\mathrm{DE}}$, $\mathrm{Q} = \overleftrightarrow{\mathrm{BC}} \cap \overleftrightarrow{\mathrm{EF}}$, $\mathrm{R} = \overleftrightarrow{\mathrm{CD}} \cap \overleftrightarrow{\mathrm{FA}}$ lie in a straight line.

It is not known how Pascal proved the theorem, or how he hit upon it (which he did at the age of sixteen). A book he was writing on the conic sections circulated for some years among the prominent mathematicians of Europe, in draft copy, and then was lost to mankind.

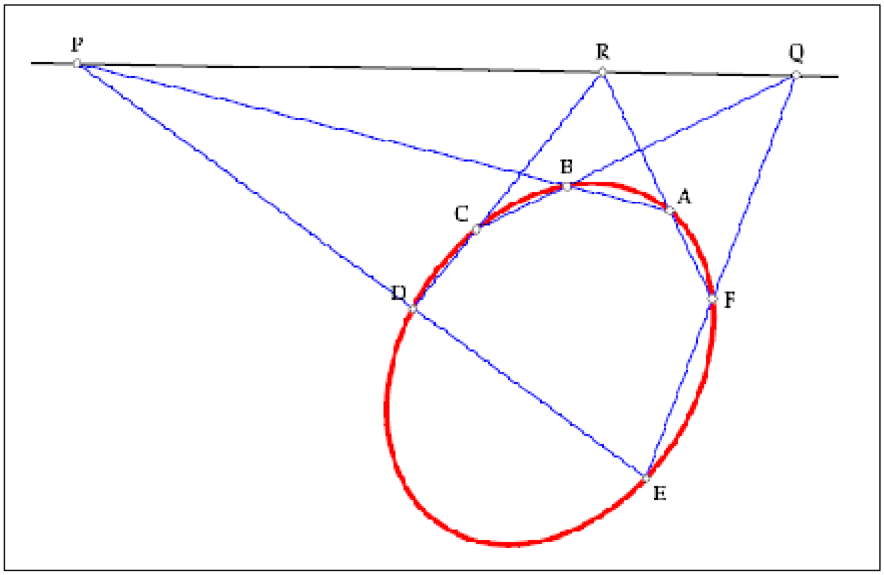

Figure 5.

Pascal's theorem is a theorem of projective geometry: it allows the points P, Q, R to lie on the 'line at infinity'. Thus, if $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$, then P is 'a point at infinity'. In this case the theorem implies that $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$. If it happens that $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$ as well as $\overleftrightarrow{BC} \parallel \overleftrightarrow{EF}$, then the theorem asserts that $\overleftrightarrow{CD} \parallel \overleftrightarrow{FA}$. (Now all three of P, Q, R lie on the line at infinity.)

An important consequence of the projective viewpoint is that the line at infinity does not have any special status; it is treated on par with every other line. So in the projective proof, it is of no consequence if some pairs of lines are parallel to each other; the wording of the proof remains exactly the same. To implement this approach, we use *projective coordinates*, in which points are denoted using triples $[x, y, z]$ of real numbers. Here are the basic rules governing these triples: (i) x, y, z are not all zero; (ii) $[kx, ky, kz]$ denotes the same point as $[x, y, z]$ for any real number $k \neq 0$. The understanding is that if $z \neq 0$, then $[x, y, z]$ corresponds to the point with cartesian coordinates $(x/z, y/z)$, and if $z = 0$ then $[x, y, z]$ lies on the line at infinity. The line with cartesian equation $x + y = 1$ acquires the equation $x + y - z = 0$ in this system, while the circle with cartesian equation

An important consequence of the projective viewpoint is that the line at infinity does not have any special status; it is treated on par with every other line.

$x^2+y^2-x-y=1$ acquires the equation $x^2+y^2-xz-yz-z^2=0$. Note that these equations are *homogeneous*. The line at infinity has the equation $z=0$.

Proof of Pascal's Theorem.

Let $L_{\rm AB}(x,y,z)=0$, $L_{\rm BC}(x,y,z)=0$, $L_{\rm CD}(x,y,z)=0$, $\ldots$ be the equations of $\overleftrightarrow{\rm AB}$, $\overleftrightarrow{\rm BC}$, $\overleftrightarrow{\rm CD}$ $\ldots$, respectively, where $L_{\rm AB}$, $L_{\rm BC}$, $L_{\rm CD}$, $\ldots$ are linear expressions in x,y,z. Let $f(x,y,z)=0$ be the equation of the conic, where f is a polynomial in x,y,z of degree 2. The various L's and f are homogeneous expressions.

Since $\mathcal{C}$ passes through $\rm A, B, C, D$, and so do the two pair-of-straight-lines conics $\overleftrightarrow{\rm AB}\cup\overleftrightarrow{\rm CD}$ and $\overleftrightarrow{\rm AD}\cup\overleftrightarrow{\rm BC}$ (see *Figure* 6), there exist constants α,α' such that

$$f \;=\; \alpha L_{\rm AB}\cdot L_{\rm CD}+\alpha' L_{\rm AD}\cdot L_{\rm BC}. \tag{1}$$

Similarly, since $\mathcal{C}$ passes through $\rm A, E, F, D$, and so do the two conics $\overleftrightarrow{\rm AD}\cup\overleftrightarrow{\rm EF}$ and $\overleftrightarrow{\rm DE}\cup\overleftrightarrow{\rm AF}$, there exist constants β,β' such that

$$f \;=\; \beta L_{\rm AD}\cdot L_{\rm EF}+\beta' L_{\rm DE}\cdot L_{\rm AF}. \tag{2}$$

From (1) and (2) we get

$$\alpha L_{\rm AB}\cdot L_{\rm CD}+\alpha' L_{\rm AD}\cdot L_{\rm BC} \;=\; \beta L_{\rm AD}\cdot L_{\rm EF}+\beta' L_{\rm DE}\cdot L_{\rm AF},$$

$$\therefore\; L_{\rm AD}\cdot(\alpha' L_{\rm BC}-\beta L_{\rm EF}) \;=\; \beta' L_{\rm DE}\cdot L_{\rm AF}-\alpha L_{\rm AB}\cdot L_{\rm CD}. \tag{3}$$

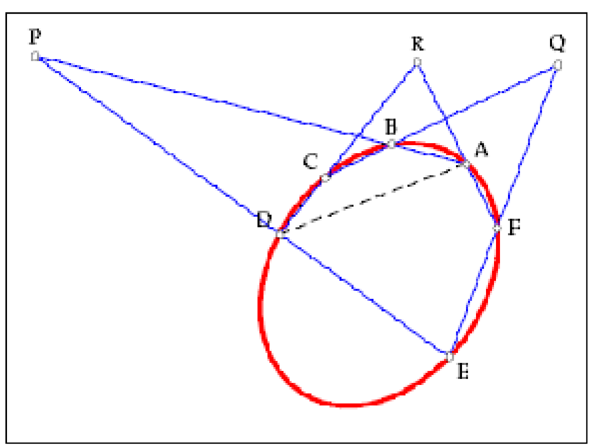

Figure 6.

Let polynomials g, h be defined as follows:

$$\begin{cases} g = \beta' L_{\mathrm{DE}} \cdot L_{\mathrm{AF}} - \alpha L_{\mathrm{AB}} \cdot L_{\mathrm{CD}}, \\ h = \alpha' L_{\mathrm{BC}} - \beta L_{\mathrm{EF}}. \end{cases} \tag{4}$$

Since it cannot happen that h is identically 0 (this would make $\overleftrightarrow{\mathrm{BC}}$ coincide with $\overleftrightarrow{\mathrm{AD}}$, which is not allowed by the hypotheses of the theorem), it must be that h has degree 1. Then the equation $h = 0$ represents a straight line. Now observe that:

When $\mathcal{C}$ is a pair of straight lines we get the result known as Pappus's theorem.

- $\mathrm{P} \in \overleftrightarrow{\mathrm{AB}}$ and $\mathrm{P} \in \overleftrightarrow{\mathrm{DE}}$, so $g(\mathrm{P}) = 0$. Since $\mathrm{P} \notin \overleftrightarrow{\mathrm{AD}}$, it follows that $h(\mathrm{P}) = 0$.
- $\mathrm{R} \in \overleftrightarrow{\mathrm{CD}}$ and $\mathrm{R} \in \overleftrightarrow{\mathrm{AF}}$, so $g(\mathrm{R}) = 0$. Since $\mathrm{R} \notin \overleftrightarrow{\mathrm{AD}}$, it follows that $h(\mathrm{R}) = 0$.
- $\mathrm{Q} \in \overleftrightarrow{\mathrm{BC}}$ and $\mathrm{Q} \in \overleftrightarrow{\mathrm{EF}}$, so $h(\mathrm{Q}) = 0$.

So $\mathrm{P}, \mathrm{Q}, \mathrm{R}$ lie on the line $\alpha' L_{\mathrm{BC}} - \beta L_{\mathrm{EF}} = 0$, and are thus collinear. □

Remarks

- The proof does not consider separately the cases of parallelism. (It is not required, as per the remarks made above.)
- Pascal's theorem is extremely wide in its scope: the hexagon ABCDEF may be non-convex and/or self-intersecting, and the conic $\mathcal{C}$ itself may be any second degree locus. When $\mathcal{C}$ is a pair of straight lines we get the result known as Pappus's theorem.
- We may even allow some pairs of points to coincide; in this case we get 'limiting cases' of the theorem by replacing 'lines' by 'tangents' as needed. For example if we let A coincide with B, we get the following statement, in which t_{X} denotes the tangent to the conic at any point X on it: *If* $\mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}$ *are five distinct points on a conic* $\mathcal{C}$, *then the points* $\mathrm{P} = t_{\mathrm{B}} \cap \overleftrightarrow{\mathrm{DE}}$, $\mathrm{Q} = \overleftrightarrow{\mathrm{BC}} \cap \overleftrightarrow{\mathrm{EF}}$, $\mathrm{R} = \overleftrightarrow{\mathrm{CD}} \cap \overleftrightarrow{\mathrm{EF}}$ *lie in a straight line.*

If ABCDEF is a hexagon formed by six lines that are all tangent to a conic, then the lines AD, BE, CF concur.

- A similar proof may be devised for Brianchon's theorem: *If* ABCDEF *is a hexagon formed by six lines that are all tangent to a conic, then the lines* $\overleftrightarrow{\mathrm{AD}}$, $\overleftrightarrow{\mathrm{BE}}$, $\overleftrightarrow{\mathrm{CF}}$ *concur.*

4. The Butterfly Theorem

The butterfly theorem first appeared as a problem in the early 1800's, and one of the early proofs is due to W G Horner who is better known for an algorithm in polynomial arithmetic.

Theorem 3 (Butterfly Theorem)**.** *Let* $\overline{\mathrm{PQ}}$ *be a chord of circle* $\mathcal{K}$*, and let* M *be its midpoint. Through* M *two other chords* $\overline{\mathrm{AB}}$ *and* $\overline{\mathrm{CD}}$ *are drawn. Let* $\overline{\mathrm{AD}}$ *and* $\overline{\mathrm{BC}}$ *cut* $\overleftrightarrow{\mathrm{PQ}}$ *in* E *and* F*, respectively. Then* M *is the midpoint of* $\overline{\mathrm{EF}}$. (See *Figure* 7.)

A large number of elegant proofs of the Butterfly Theorem have appeared over the years, including many 'pure geometry' proofs; but the following proof is particularly charming, for it simultaneously casts the theorem in a more general setting and makes it easy to formulate special cases of interest. It is based yet again on the assertions made in Section 2.

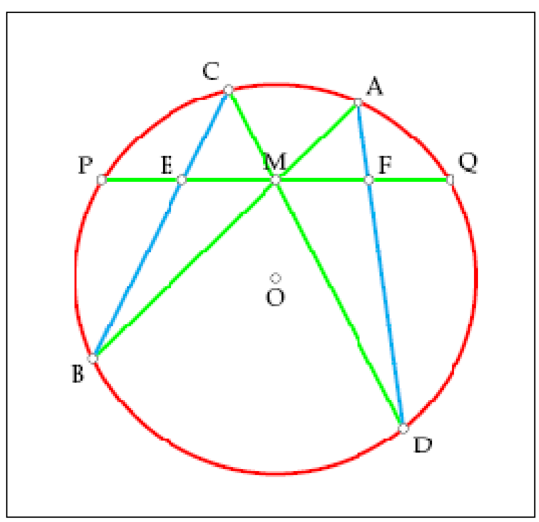

Figure 7.

Theorem 4 (Butterfly Theorem for Conics)**.** *Let* $\overline{\mathrm{PQ}}$ *be a chord of a conic* $\mathcal{K}$*, and let* M *be its midpoint. Through* M *let two other chords* $\overline{\mathrm{AB}}$ *and* $\overline{\mathrm{CD}}$ *be drawn. Let* $\overline{\mathrm{AD}}$ *and* $\overline{\mathrm{BC}}$ *cut* $\overleftrightarrow{\mathrm{PQ}}$ *in* E *and* F*, respectively. Then* M *is the midpoint of* $\overline{\mathrm{EF}}$*.* (See *Figure* 8.)

Proof. We use a cartesian setting, with $\overleftrightarrow{\mathrm{PQ}}$ as the x-axis, and M as the origin. Let $\mathcal{K}$ have equation $p(x,y) = 0$, where $p(x,y) = ax^2 + by^2 + cxy + dx + ey + f$. The intersections of $\mathcal{K}$ with the x-axis are found by solving the equations $p(x,y) = 0$, $y = 0$, i.e.,

$$ax^2 + dx + f = 0. \qquad (5)$$

Since M is the midpoint of $\overline{\mathrm{PQ}}$, the sum of the roots of (5) is 0, which implies that $d = 0$. *Hence, the coefficient of the* x*-term in* $p(x,y)$ *is zero.*

This assertion also holds for the line-pair conic $\mathcal{K}' = \overleftrightarrow{\mathrm{AB}} \cup \overleftrightarrow{\mathrm{CD}}$, because the lines $\overleftrightarrow{\mathrm{AB}}$ and $\overleftrightarrow{\mathrm{CD}}$ pass through the origin. *That is, if the equation of* $\mathcal{K}'$ *is* $q(x,y) = 0$*, then the coefficient of the* x*-term in* $q(x,y)$ *is zero.*

Since $\mathcal{K}$ and $\mathcal{K}'$ share the four points $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$, *the same statement is true for any conic that passes through* $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$. This is because the equation of any such conic is of the form $rp(x,y) + sq(x,y) = 0$, where r, s are real numbers.

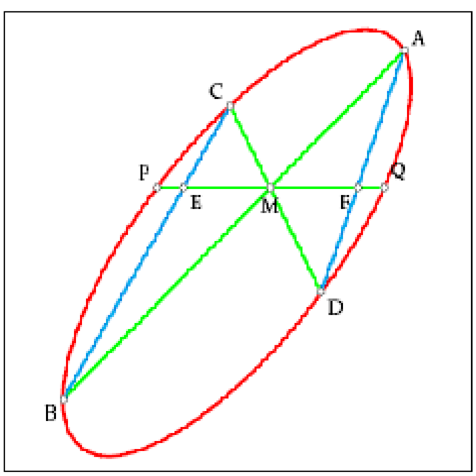

Figure 8.

A surprising consequence of Pascal's theorem is that it allows us to define a group on the points of any non-degenerate conic.

In particular it is true for the line-pair conic $\mathcal{K}'' = \overleftrightarrow{\mathrm{AC}} \cup \overleftrightarrow{\mathrm{BD}}$. Hence, the sum of the roots of the intersections of $\mathcal{K}''$ with the x-axis is 0. That is, M is the midpoint of $\overline{EF}$. □

Here are two typical 'special cases' whose proofs we leave to the reader:

Theorem 5. *Let* ABCD *be a cyclic quadrilateral whose circumcircle* $\mathcal{K}$ *has* $\overline{\mathrm{AC}}$ *as a diameter, and* O *as its center. Let the tangent to* $\mathcal{K}$ *at* A *meet* $\overleftrightarrow{\mathrm{BD}}$ *at* P*; let* $\overleftrightarrow{\mathrm{PO}}$ *meet* $\overleftrightarrow{\mathrm{CB}}$ *in* E*, and* $\overleftrightarrow{\mathrm{CD}}$ *in* F*, respectively. Then* O *is the midpoint of* $\overline{\mathrm{EF}}$. (See *Figure* 9.)

Theorem 6. *Let* $\overline{\mathrm{AB}}$, $\overline{\mathrm{CD}}$ *be segments intersecting at a point* P*, and let* Q *and* R *be points on* $\overline{\mathrm{AB}}$ *and* $\overline{\mathrm{CD}}$*, respectively, such that* $\overline{\mathrm{AP}} = \overline{\mathrm{QB}}$*, and* $\overline{\mathrm{CP}} = \overline{\mathrm{RD}}$*. Let* $\overleftrightarrow{\mathrm{QR}}$ *meet* $\overline{\mathrm{AD}}$ *in* U*, and* $\overline{\mathrm{BC}}$ *in* V*. Then* $\overline{\mathrm{UR}} = \overline{\mathrm{QV}}$. (See *Figure* 10.)

5. Group on a Conic

In closing we point out a rather surprising consequence of Pascal's theorem: it allows us to define a group on the points of any non-degenerate conic. Let N be any fixed point of such a conic $\mathcal{K}$; this will serve as the neutral point, i.e., the identity element of the group. We define

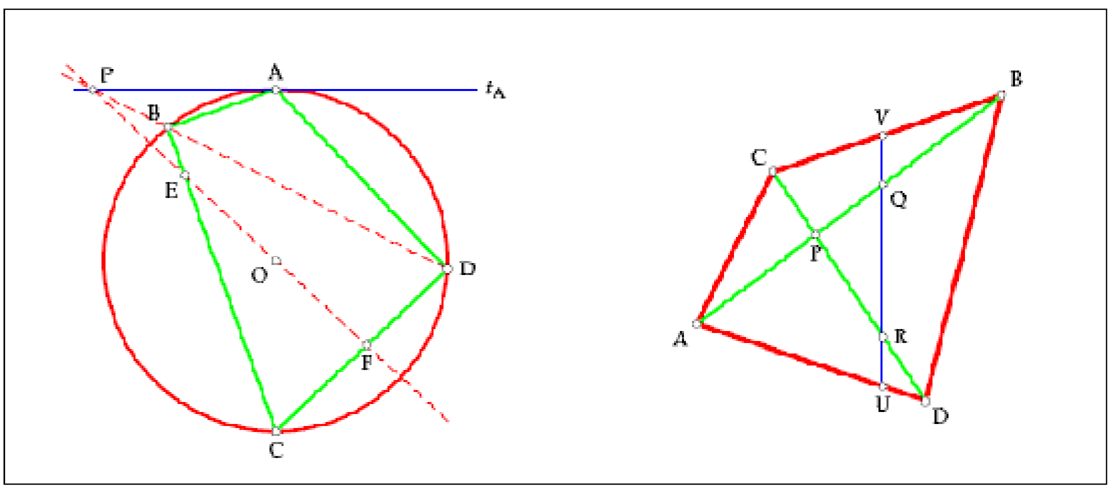

Figure 9 (left).
Figure 10 (right).

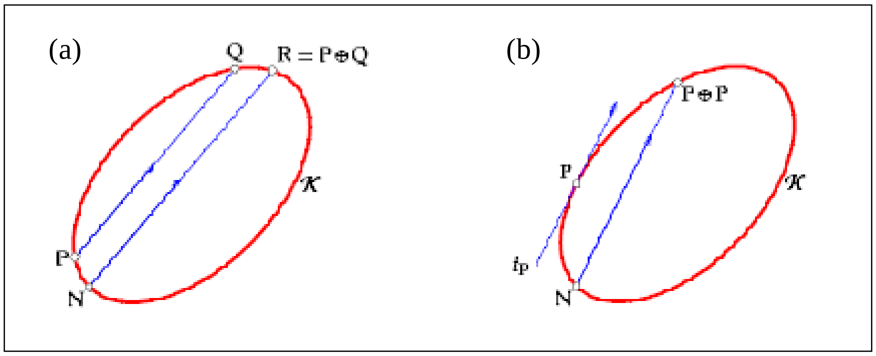

Figure 11.

the binary operation $\oplus$ thus: if P, Q are points on $\mathcal{K}$, we draw through N a line parallel to $\overleftrightarrow{\mathrm{PQ}}$; then $\mathrm{P} \oplus \mathrm{Q}$ is the point where this line intersects $\mathcal{K}$ again. The construction is as depicted in *Figure* 11a.

It is easy to check the following assertions:

- The operation is well defined. For, the line through N parallel to $\overleftrightarrow{\mathrm{PQ}}$ will intersect $\mathcal{K}$ a second time, since $\mathcal{K}$ is a second degree curve.
- $\mathrm{P} \oplus \mathrm{Q} = \mathrm{Q} \oplus \mathrm{P}$ for all pairs of points $\mathrm{P}, \mathrm{Q} \in \mathcal{K}$; so $\oplus$ is commutative.
- If $\mathrm{P} = \mathrm{Q}$, then $\overleftrightarrow{\mathrm{PQ}}$ is tangent to $\mathcal{K}$ at P. So to find $\mathrm{P} \oplus \mathrm{P}$, we draw a line through N parallel to the tangent t_{P} to $\mathcal{K}$ at P; then its second point of intersection with $\mathcal{K}$ is $\mathrm{P} \oplus \mathrm{P}$. See *Figure* 11b.
- We have $\mathrm{N} \oplus \mathrm{N} = \mathrm{N}$, and $\mathrm{P} \oplus \mathrm{N} = \mathrm{P}$ for all $\mathrm{P} \in \mathcal{K}$.
- If the line through N parallel to $\overleftrightarrow{\mathrm{PQ}}$ is tangent to $\mathcal{K}$ at N, then $\mathrm{R} = \mathrm{N}$, so we write $\mathrm{P} \oplus \mathrm{Q} = \mathrm{N}$. This enables us to find inverses.

If we invoke Pascal's theorem in this configuration we find fairly easily that $\mathrm{P} \oplus (\mathrm{Q} \oplus \mathrm{R}) = (\mathrm{P} \oplus \mathrm{Q}) \oplus \mathrm{R}$ for any three points $\mathrm{P}, \mathrm{Q}, \mathrm{R}$ on $\mathcal{K}$. This implies the following result.

Theorem 7. *The pair* $(\mathcal{K}, \oplus)$ *is an abelian group, with* N *as the identity element.*

It is interesting to classify the groups corresponding to different conics. Here are the main results, whose proofs we leave to the reader:

1. If $\mathcal{K}$ is either an ellipse or a circle, then $(\mathcal{K}, \oplus)$ is isomorphic to the multiplicative group of complex numbers with unit magnitude, i.e., isomorphic to the group $\mathbb{R}/2\pi\mathbb{Z}$.
2. If $\mathcal{K}$ is a parabola, then $(\mathcal{K}, \oplus)$ is isomorphic to the additive group of real numbers, $(\mathbb{R}, +)$.
3. If $\mathcal{K}$ is a hyperbola, then $(\mathcal{K}, \oplus)$ is isomorphic to the multiplicative group of non-zero real numbers, $(\mathbb{R}^\star, \times)$.

There are numerous questions of interest which can be explored in this regard, including some which are of a number-theoretic nature. These arise when one regards the conic as defined over some finite field rather than the field of real numbers; e.g., over the integers modulo p for some prime p. The conic in such a case is not a visualizable object, and one has to study it algebraically rather than geometrically. Here is a typical result: *If $\mathcal{K}$ is the conic $x^2-2y^2 = 1$ over the field of integers modulo a prime p, then $(\mathcal{K}, \oplus)$ is a cyclic group; its order is $p-1$ if 2 is a quadratic residue modulo p, and $p+1$ if 2 is a quadratic nonresidue modulo p.* However we shall not dwell on this topic here. The reader is referred to [**4**] for further details.

Suggested Reading

[1] **H S M Coxeter, *Introduction to Geometry*, 2nd edition, Wiley, 1989.**

[2] **H S M Coxeter and S L Greitzer, *Geometry Revisited*, Math. Assoc. Amer., 1st edition, 1967.**

[3] **S L Loney, *The Elements of Coordinate Geometry*, in two volumes, MacMillan India, 2000.**

[4] **Shailesh Shirali, *Groups Associated With Conics*, The Mathematical Gazette, (to appear February 2009).**

[5] **Eric W Weisstein, *Conic Section*, From MathWorld – A Wolfram Web Resource, http://mathworld.wolfram.com/ConicSection.html**

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