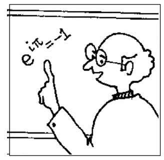
Classroom



In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

M P Chaudhary
International Scientific Research & Welfare Organization, Vikaspuri
New Delhi 110018, India.
Email: mpchaudhary_2000@yahoo.com

Simple Solution of Some Integrals given by Srinivasa Ramanujan

Srinivasa Ramanujan (1887–1920) had generated several research problems in the area of number theory. However, they were scattered for a long time. Finally, in 1957, all his results were compiled and published in four volumes (known as *Lost Note Books* of Ramanujan) by Tata Institute of Fundamental Research, Bombay. Since Ramanujan’s death, several mathematicians have proposed different solutions of the various problems given by him.

Here we give simple solutions of two integrals (currently existing as entry VII in chapter XVII of the second note book).

Problem 1.

$$\text{If} \quad \frac{\sin\alpha}{\sin\beta} = \sqrt{\frac{1-x}{x}}, \tag{1.1}$$

then show that

$$\int_0^{\alpha} \frac{\mathrm{d}\theta}{\sqrt{1 - x\cos^2\theta}} = \int_0^{\beta} \frac{\mathrm{d}\theta}{\sqrt{x - \sin^2\theta}}. \qquad (1.2)$$

Solution: Let I denote the LHS if (1.2). That is,

$$\mathrm{I} = \int_0^{\alpha} \frac{\mathrm{d}\theta}{\sqrt{1 - x\cos^2\theta}}. \qquad (1.3)$$

From equation (1.1) we have

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{\frac{1-x}{x}}, \text{or } \alpha = \sin^{-1}\left\{\sqrt{\frac{1-x}{x}}\sin\beta\right\}. \quad (1.4)$$

Substituting this in (1.3), we get

$$\mathrm{I} = \int_{\theta=0}^{\theta=\sin^{-1}\left\{\sqrt{\frac{1-x}{x}}\sin\beta\right\}} \frac{\mathrm{d}\theta}{\sqrt{1 - x\cos^2\theta}}. \qquad (1.5)$$

Now using $\sin\theta = \sqrt{\frac{1-x}{x}}.\tan t$ in (1.4), we get

$$\mathrm{I} = \int_{\theta=0}^{\theta=\alpha} \frac{\sqrt{1-x}\sec^2 t\,\mathrm{d}t}{\sqrt{x\sec^2 t - \tan^2 t}\sqrt{(1-x) + (1-x)\tan^2 t}}$$

On simplifying,

$$\mathrm{I} = \int_0^{\beta} \frac{\mathrm{d}t}{\sqrt{x - \sin^2 t}}$$

$$\text{or, } \mathrm{I} = \int_0^{\beta} \frac{\mathrm{d}\theta}{\sqrt{x - \sin^2\theta}} \qquad (1.6)$$

From equations (1.3) and (1.6), we get

$$\int_0^{\alpha} \frac{\mathrm{d}\theta}{\sqrt{1 - x\cos^2\theta}} = \int_0^{\beta} \frac{\mathrm{d}\theta}{\sqrt{x - \sin^2\theta}},$$

which is the required result.

Problem 2.

$$\text{If } \frac{\sin\alpha}{\sin\beta} = \sqrt{1 - b\cos^2\alpha} \quad \text{or} \quad \frac{\tan\alpha}{\tan\beta} = \sqrt{1-b}, \quad (2.1)$$

then show that

$$\int_0^\beta \frac{\mathrm{d}\theta}{\sqrt{(1-a\sin^2\theta)(1-b\sin^2\theta)}} = \frac{1}{\sqrt{1-b}} \int_0^\alpha \frac{\mathrm{d}\theta}{\sqrt{1-\left(\frac{a-b}{1-b}\right)\sin^2\theta}}. \quad (2.2)$$

Solution: Let J denote the LHS of equation (2.2). That is,

$$\mathrm{J} = \int_0^\beta \frac{\mathrm{d}\theta}{\sqrt{(1-a\sin^2\theta)(1-b\sin^2\theta)}}. \quad (2.3)$$

Putting $\sin\theta = \frac{\sin t}{\sqrt{(1-b\cos^2 t)}}$ in equation (2.3), we get

$$\mathrm{J} = \int_0^\alpha \frac{\cos t(1-b)\mathrm{d}t}{\mathrm{D}}, \text{where}$$

$$\mathrm{D} = (1-b\cos^2 t)\sqrt{1-b\cos^2 t - \sin^2 t}$$

$$\times \left\{\frac{1-b\cos^2 t - a\sin^2 t}{1-b\cos^2 t}\right\}^{1/2} \left\{\frac{1-b}{1-b\cos^2 t}\right\}^{1/2}. \quad (2.4)$$

On simplifying

$$\mathrm{J} = \frac{1}{\sqrt{(1-b)}} \int_0^\alpha \frac{\mathrm{d}t}{\sqrt{1-\left(\frac{a-b}{1-b}\right)\sin^2 t}}. \quad (2.5)$$

From equations (2.3) and (2.5), we get

$$\int_0^\beta \frac{\mathrm{d}\theta}{\sqrt{(1-a\sin^2\theta)(1-b\sin^2\theta)}} = \frac{1}{\sqrt{1-b}} \int_0^\alpha \frac{\mathrm{d}\theta}{\sqrt{1-\left(\frac{a-b}{1-b}\right)\sin^2\theta}}.$$

This is the required result.