Snippets of Physics

9. Ambiguities in Fluid Flow

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The idealized flow of fluid around a spherical body is a classic textbook problem in fluid mechanics. Interestingly enough, it leads to some curious twists and turns and conceptual conundrums.

The flow induced in a fluid when a body moves through it is of tremendous practical importance – with the airplane wings providing just one example. In general, nobody understands the flow of *real* fluids and we have to resort to scaled models (e.g., in wind tunnels) or to numerical simulations to make progress. But there are some *idealized* models that one can solve analytically which corresponds, broadly speaking, to the mythical fluid sometimes called "dry water". These problems are supposedly well-understood and we will see, in this installment, that even the simplest of them can lead to surprises.

When a body moves through the hypothetical fluid, the resulting flow satisfies the following conditions: First, the fluid flow is incompressible with the density being a constant. Then, the conservation of mass, expressed in the form of a continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad (1)$$

(in which ρ is the density and $\boldsymbol{v}$ is the fluid velocity) reduces to the simple condition $\nabla \cdot \boldsymbol{v} = 0$. Second, we will assume that the flow is irrotational ($\nabla \times \boldsymbol{v} = 0$) allowing for the velocity to be expressed as a gradient of a scalar potential $\boldsymbol{v} = \nabla\phi$. Finally we will ignore all the properties of real fluids like viscosity, surface tension,

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etc., and will treat the problem as one of finding the solutions to the two equations $\nabla \cdot \boldsymbol{v} = 0$ and $\nabla \times \boldsymbol{v} = 0$ subject to certain boundary conditions. Equivalently, we find that the potential satisfies Laplace's equation $\nabla^2 \phi = 0$. So the problem reduces to solving the Laplace equation with $\boldsymbol{v}$ satisfying the boundary conditions – which is the only nontrivial feature of the problem!

Let us consider a body of an arbitrary shape moving with a velocity $\boldsymbol{u}$ though the fluid. Then we need to solve the Laplace equation subject to the boundary condition $\boldsymbol{n} \cdot \boldsymbol{v} = \boldsymbol{n} \cdot \boldsymbol{u}$ at the surface, where $\boldsymbol{n}$ is the normal to the surface. We would expect the fluid flow near the body to be affected by its motion, but at sufficiently large distance this effect should be negligible. Hence the fluid velocity $\boldsymbol{v}$ will be zero at spatial infinity.

Interestingly enough the general form of the fluid velocity at large distances from the body (of arbitrary shape) can be determined by the following argument. We know that the function $1/r$ satisfies the Laplace equation. Further, if ϕ satisfies the Laplace equation, the spatial derivatives of ϕ also satisfy the same equation. Therefore, the directional derivative of $1/r$, along some direction specified by an arbitrary vector $\boldsymbol{A}$ will also satisfy the Laplace equation. Such a directional derivative is given by $\boldsymbol{A} \cdot \nabla(1/r)$ and will fall as $1/r^2$ at large distances. Hence, at large distances from the body, we can take the leading order terms in the potential to be

$$\phi = -\frac{q}{r} + \boldsymbol{A} \cdot \nabla \left(\frac{1}{r}\right) + \mathcal{O}(1/r^3). \qquad (2)$$

You will, of course recognize all these to be electrostatics in disguise and the expansion in (2) to be just the large distance expansion of the potential due to the distribution of charges. The first term is the monopole coulomb term and the second one is the dipole term. (Incidentally, the dipole term is just the difference in the potential due to two charges kept separated by a distance $\boldsymbol{A}$; clearly, the net potential will be the directional derivative along $\boldsymbol{A}$. This is the quickest way to get the dipole potential.) At sufficiently large distances we ignore further terms, obtained by taking the second, third, ... derivatives of $1/r$.

The velocity field will then be the analogue of the electric field in electrostatics. From Gauss' law we know that the flux of the electric field at large distances is proportional to the 'total charge' q. Since we cannot have a non-zero flux of velocity at large distance in our problem, it follows that $q = 0$ and the asymptotic form of the potential must have the form:

$$\phi = \boldsymbol{A} \cdot \nabla \left(\frac{1}{r}\right) = -\frac{\boldsymbol{A} \cdot \boldsymbol{n}}{r^2}, \qquad (3)$$

where $\boldsymbol{n}$ is the unit vector in the radial direction. Taking the gradient, we get the velocity field to be

$$\boldsymbol{v} = (\boldsymbol{A} \cdot \nabla)\nabla \left(\frac{1}{r}\right) = \frac{3(\boldsymbol{A} \cdot \boldsymbol{n})\boldsymbol{n} - \boldsymbol{A}}{r^3}. \qquad (4)$$

(These manipulations are most efficiently done using index notation and summation convention, with $\partial_\alpha r = (1/2r)\partial_\alpha r^2 = x^\alpha/r$ used repeatedly.) The actual form of $\boldsymbol{A}$ needs to be determined using the conditions near the body (which will be a mess for a body of arbitrary shape) but it is interesting that the flow at large distances is fixed entirely in terms of a single vector $\boldsymbol{A}$. In fluid mechanics, it is a bit of a surprise but in electrostatics it is not. If the monopole vanishes, you would expect the dipole moment to determine the behaviour of the electric field at large distances.

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The real surprise comes when we try to calculate the total kinetic energy associated with the fluid flow given by

$$K_{\text{lab}} = \frac{1}{2}\rho \int \mathrm{d}^3\boldsymbol{x}\ v^2, \qquad (5)$$

where the integral is over all space outside the body and the subscript 'lab' stands for the lab frame in which the body is moving with a velocity $\boldsymbol{u}$. (The fact that the body is moving is irrelevant since it only shifts the origin by $\boldsymbol{u}t$ which is a constant as far as the spatial integration is concerned.) While the fluid flow at large distances can be expressed entirely in terms of a single vector $\boldsymbol{A}$, the flow closer to the body can be extremely complicated. One might have thought that, in such a general case, one cannot say anything about the total kinetic energy of the fluid. But it is indeed possible to express the total kinetic energy of the fluid flow entirely in terms of the single vector $\boldsymbol{A}$ even though the fluid flow everywhere cannot be expressed in terms of $\boldsymbol{A}$ alone. (This result, as well as equations (8) and (18) below, are derived in [1] but not discussed in detail in any other book, as far as I know.)

To obtain this result, we use the identity $v^2 = u^2 + (\boldsymbol{v} + \boldsymbol{u}) \cdot (\boldsymbol{v} - \boldsymbol{u})$. If we integrate both sides of this equation over a large volume V, the first term on the right will give a contribution proportional to $(V - V_{\text{body}})$. In the second term, we write $(\boldsymbol{v}+\boldsymbol{u}) = \nabla(\phi+\boldsymbol{u}\cdot\boldsymbol{r})$. Now using $\nabla \cdot \boldsymbol{v} = 0$, and $\nabla \cdot \boldsymbol{u} = 0$, we can write the second term as a total divergence $\nabla \cdot [(\phi + \boldsymbol{u} \cdot \boldsymbol{r})(\boldsymbol{v} - \boldsymbol{u})]$. Integrating this over the whole space, the second term becomes a surface integral over both the surface of the body and a surface at large distance. That is, we have proved:

$$\int v^2 \mathrm{d}V = u^2(V - V_0) + \oint_{S+S_0} (\phi + \boldsymbol{u} \cdot \boldsymbol{r})(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{6}$$

where S is a surface bounding the volume V at large distance and S_0 is the surface of the body. The surface integral is taken over both. The miracle is now in sight. On the surface of the body, $(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{n}$ vanishes due to the boundary conditions and hence we get no contributions from there! This is good since we have no clue about the pattern of velocity flow near the body. At large distances from the body, we can use the asymptotic form of the velocity field given in (4) and do the integral taking the surface to be a sphere of large radius R. Since $\mathrm{d}S = R^2\mathrm{d}\Omega$ increases as R^2 while v falls as $1/R^3$ and ϕ falls as $1/R^2$ we can approximate $\phi(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{n} \approx -\phi\boldsymbol{u}\cdot\boldsymbol{n}$ on S. Hence the surface integral in (6) on S becomes the sum

$$-\oint_S \phi\boldsymbol{u}\cdot\boldsymbol{n}\; R^2\mathrm{d}\Omega \;+\; \oint_S (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n})\; R^3\mathrm{d}\Omega - \oint_S (\boldsymbol{u}\cdot\boldsymbol{n})^2\; R^3\mathrm{d}\Omega. \qquad (7)$$

The integration over angular coordinates can be done using the easily proved relation $<(\boldsymbol{A}\cdot\boldsymbol{n})(\boldsymbol{B}\cdot\boldsymbol{n})>= (1/3)\boldsymbol{A}\cdot\boldsymbol{B}$ where $< ... >$ denotes the angular *average* which is $1/4\pi$ times the integral over $\mathrm{d}\Omega$. Using this, we see that the integral over $-(\boldsymbol{u}\cdot\boldsymbol{n})^2R^3$ gives $-u^2V$ which precisely cancels the u^2V in the first term in (6). Using (3) and (4) we get the final answer to be:

$$K_{\mathrm{lab}} = \frac{1}{2}\,\rho\,(4\pi\boldsymbol{A}\cdot\boldsymbol{u} - V_0u^2). \qquad (8)$$

Thus, if we know the motion of the fluid at very large distances from the body, we can compute the total kinetic energy of the fluid flow without ever knowing the velocity field close to the body![1]

[1] The electrostatic analogue is the following. You are given a distribution of charges with $q_{\mathrm{tot}} = 0$ and dipole moment $\boldsymbol{p}$ in a region bounded by a surface S_0. You are also given a constant vector $\boldsymbol{E}_0$ and told that the normal component of the electric field normal to S_0 is given by $\boldsymbol{n}\cdot\boldsymbol{E}_0$. Then the electrostatic energy is proportional to $(4\pi\boldsymbol{p}\cdot\boldsymbol{E}_0 - V_0E_0^2)$ where V_0 is the volume of the region bounded by S_0.

One can use this to obtain another curious result. To do this, we note that the K_{lab} can also be expressed in a different form of surface integral. Writing $\boldsymbol{v} = \nabla\phi$, the expression for kinetic energy reduces to

$$K = \frac{1}{2}\rho\int_{\mathcal{V}} \mathrm{d}^3\boldsymbol{x}(\nabla\phi)^2 = \frac{1}{2}\rho\int_{\mathcal{V}} \mathrm{d}^3\boldsymbol{x}\nabla\cdot(\phi\nabla\phi), \qquad (9)$$

where we have used $\nabla^2\phi = 0$. Using Gauss' theorem, this expression can be converted to a surface integral over the body and over a surface at large distance. The second one vanishes, giving

$$K_{\rm lab} = -\frac{1}{2}\rho \oint_{S_0} {\rm d}S(\boldsymbol{n}\cdot\boldsymbol{v})\phi = -\frac{1}{2}\rho \oint_{S_0} {\rm d}S(\boldsymbol{n}\cdot\boldsymbol{u})\phi, \tag{10}$$

where we have used $\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{u}$ at the surface. But since we know what $K_{\rm lab}$ is, this allows us to obtain an integral over the surface of the body of a particular expression, in the form

$$-\oint_{S_0} {\rm d}S(\boldsymbol{n}\cdot\boldsymbol{u})\phi = (4\pi\boldsymbol{A}\cdot\boldsymbol{u} - V_0 u^2) \tag{11}$$

even though we do not know either the shape of the body or the velocity potential on the surface!

Let us now specialize to the simplest of all possible shapes for the body: a sphere of radius a. In this case, the dipole potential happens to be the *exact* solution at all distances outside the sphere. This is not difficult to understand. Given the spherical symmetry, the only vector that can appear in the solution is the velocity of the body $\boldsymbol{u}$. Linearity of the Laplace equation (and the boundary condition) tells us that the potential must be linear in this vector $\boldsymbol{u}$. Hence the solution must have the form in equation (3) with $\boldsymbol{A} \propto \boldsymbol{u}$. Using the boundary condition $\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{u}$ at the surface, it is easy to show that

$$\boldsymbol{A} = \frac{1}{2}a^3\boldsymbol{u} \tag{12}$$

which completely solves the problem. We will now play around with this solution.

Given the fluid flow pattern everywhere, we can explicitly compute the total kinetic energy carried by the flow using any of the expressions derived above. We get

$$K_{\rm lab} = -\frac{1}{2}\rho \int a^2 {\rm d}\Omega \left(-\frac{1}{a^2}\right)(\boldsymbol{A}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n})$$
$$= \frac{1}{2}\rho(4\pi)\frac{1}{3}(\boldsymbol{A}\cdot\boldsymbol{u}) = \frac{1}{4}m_{\rm disp}u^2, \qquad (13)$$

where $m_{\rm disp}$ is the mass of the fluid displaced by the sphere. So the total kinetic energy is $(1/2)[m_{\rm body} + (1/2)m_{\rm disp}]u^2$ and the fluid adds $(1/2)m_{\rm disp}$ to the effective mass of the sphere. Of course, our general expression, equation (8) leads to the same result when we use (12) and everything seems fine.

Let us next consider the total momentum $\boldsymbol{P}$ carried by the fluid which is the integral over all space of $\rho\boldsymbol{v}$. In a reasonable world, we would have expected it to be $(1/2)m_{\rm disp}\boldsymbol{u}$ but we are in for a rude shock. By symmetry, the vector $\boldsymbol{P}$ has to be in the direction of $\boldsymbol{u}$ so we only need to compute the scalar $\boldsymbol{P}\cdot\boldsymbol{u}$. But since v falls as $1/r^3$ and the volume grows as r^3 we are in trouble! (This did not happen for the kinetic energy since we were integrating $v^2 \propto 1/r^6$ over all space.). Explicitly, we have,

$$\boldsymbol{P}_{\rm lab}\cdot\boldsymbol{u} = \rho\int {\rm d}^3\boldsymbol{x}\,\frac{1}{r^3}\left[3(\boldsymbol{A}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n}) - \boldsymbol{A}\cdot\boldsymbol{u}\right]$$
$$= \rho\int_a^\infty \frac{{\rm d}r}{r}\int {\rm d}\Omega\left[3(\boldsymbol{A}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n}) - \boldsymbol{A}\cdot\boldsymbol{u}\right]. \qquad (14)$$

Whoever would have guessed that the simplest problem in fluid flow past a body will actually lead to a product of zero and infinity!

Obviously, our power counting argument is correct and the r-integral diverges logarithmically at large distances! On the other hand the angular integration over spherical surfaces gives zero because $< 3(\boldsymbol{A}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n}) >= \boldsymbol{A}\cdot\boldsymbol{u}$ cancels the second term. Whoever would have guessed that the simplest problem in fluid flow past a body will actually lead to a product of zero and infinity!

If we do the integral between two spheres of radii $r = a$ and $r = R$ centred on the moving sphere at any given instant of time, then the answer is indeed zero because the angular average gives zero. This would have been an acceptable result, except for two reasons. First, the result depends on taking the outer boundary to be a sphere. If we choose some other shape, say, a cylinder coaxial with the direction of motion of the sphere, the result can be different. One feels uneasy about the result being dependent on what one is doing at infinity especially since the direction of $\boldsymbol{u}$ breaks the spherical symmetry.

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Second, one can argue that, if you push the sphere from rest to let it acquire a velocity $\boldsymbol{u}$, then – in the process – you impart some momentum to the fluid. To do this computation one needs to know the pressure which acts on the sphere when $\boldsymbol{u}$ is a function of time [2]. Let me briefly indicate how this can be obtained. The starting point is the Euler equation $\partial \boldsymbol{v}/\partial t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla p/\rho$. When $\boldsymbol{v} = \nabla\phi(t, \boldsymbol{x})$, you can manipulate this equation to show that $\nabla[p + (1/2)\rho v^2 + \rho(\partial\phi/\partial t)] = 0$, so that the pressure can be expressed in the form

$$p = p_\infty - \frac{1}{2}\rho v^2 - \rho\frac{\partial\phi}{\partial t}, \tag{15}$$

where p_∞ is the pressure at infinity. (This is just a time dependent version of Bernoulli's equation.) We are interested in the net force in the direction of motion of the sphere, taken to be the z-axis, which can be obtained by integrating $p\cos\theta$ over the surface of the sphere. From (4) we see that v^2 will be a function of $\cos^2\theta$ so the contribution from the first two terms will vanish on integration over a sphere. The only surviving contribution comes from the last term which can be easily evaluated to give

$$F_z = -\int_0^\pi 2\pi a^2 \sin\theta \mathrm{d}\theta \left[\frac{1}{2}\rho a \cos^2\theta \frac{\mathrm{d}u_z}{\mathrm{d}t}\right] = \frac{1}{2} m_{\mathrm{disp}} \frac{\mathrm{d}u_z}{\mathrm{d}t}. \tag{16}$$

Clearly, the total momentum imparted is

$$\int F_z \mathrm{d}t = \frac{1}{2} m_{\mathrm{disp}} u_z, \tag{17}$$

which makes sense when we remember that the kinetic energy comes with the effective mass $(1/2)m_{\mathrm{disp}}$. So, this is another purely local reason to believe the total momentum of the fluid flow is non-zero.

In fact, one can generalize this argument and obtain a finite expression for the momentum for any body moving through a fluid [1]. Once again the result can be expressed entirely in terms of the vector $\boldsymbol{A}$ for a body of *arbitrary* shape. To obtain this result, we can use a trick which relates the infinitesimal changes in the energy and momentum by the relation $\mathrm{d}E = \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{P}$ and use the result in (8). To prove this relation, let us assume that the body is accelerated by some external force $\boldsymbol{F}$ causing the momentum of fluid flow to increase by an amount $\mathrm{d}\boldsymbol{P}$ in a time interval $\mathrm{d}t$. From the relation $\mathrm{d}\boldsymbol{P} = \boldsymbol{F}\mathrm{d}t$ we immediately get $\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{P} = \boldsymbol{F} \cdot \boldsymbol{u}\mathrm{d}t = \mathrm{d}E$. Given the form of E, it is now an elementary matter to verify that the total momentum of the fluid flow is given by

$$\boldsymbol{P} = 4\pi\, \rho \boldsymbol{A} - \rho V_0 \boldsymbol{u}. \tag{18}$$

We see that this is, in general, non-zero. In the case of the sphere it does give $(1/2)m_{\mathrm{disp}}\boldsymbol{u}$; this is what we would have naively expected. Of course, the argument is designed to give this.

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The need for regularizing the problem by introducing a very large but finite volume for the total fluid becomes more apparent when we study the same result in the rest frame of the sphere. In this frame, we have a sphere of radius a located around the origin and the fluid is flowing past it. The boundary condition at infinity is now different and we expect the fluid velocity to reach a constant value $-\boldsymbol{u}$ at large distances. This is easily achieved by adding a constant electric field to a dipole in the electrostatic case. This leads to a velocity potential of the form

$$\psi = -\boldsymbol{r}\cdot\boldsymbol{u} + \phi = -\boldsymbol{r}\cdot\boldsymbol{u} - \frac{\boldsymbol{A}\cdot\boldsymbol{n}}{r^2}. \qquad (19)$$

We denote the velocity potential in the rest frame by ψ to distinguish it from the velocity potential in the lab frame ϕ. Let us now ask what is the kinetic energy of the fluid in this frame in which the body is at rest. The fluid velocity now is $\boldsymbol{V} = \boldsymbol{v} - \boldsymbol{u}$. The kinetic energy in the rest frame will be

$$\begin{aligned} K_{\rm rest} &= \int \mathrm{d}^3\boldsymbol{x}\frac{1}{2}\rho\; V^2 = \frac{1}{2}\rho\int \mathrm{d}^3\boldsymbol{x}\left[v^2 + u^2 - 2\boldsymbol{v}\cdot\boldsymbol{u}\right] \\ &= \frac{1}{2}\rho\int \mathrm{d}^3\boldsymbol{x}\, u^2 - \boldsymbol{u}\cdot\boldsymbol{P}_{\rm lab} + K_{\rm lab}. \qquad (20) \end{aligned}$$

We see that the last term is the kinetic energy in the lab frame computed above, which is quite well-defined. The second term is ambiguous and vanishes if we use spherical regularization while is given by (18) if we use local energy conservation arguments. In the latter case, $K_{\rm lab} - \boldsymbol{u}\cdot\boldsymbol{P}_{\rm lab} = -(1/4)m_{\rm disp}u^2$ is *negative*. The first term, however, will be divergent if we take the volume of the fluid to be infinite and is positive. This divergence arises because if the fluid extends all the way to infinity then most of it will be moving with a velocity $-\boldsymbol{u}$ in the rest frame of the sphere. This will contribute an infinite amount of kinetic energy. While quite understandable, it shows that Galilean invariance needs to be used with care in the presence of an external medium.

Suggested Reading

[1] **L D Landau, E M Lifshitz, *Fluid Mechanics*, Section 10,11, Pergamon, 1989.**

[2] **T E Faber, *Fluid Mechanics for Physicists*, Section 4.8, Cambridge University Press, 1995.**

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