Snippets of Physics

8. Foucault Meets Thomas

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The Foucault pendulum is an elegant device that demonstrates the rotation of the Earth. After describing it, we will elaborate on an interesting geometrical relationship between the dynamics of the Foucault pendulum and Thomas precession discussed in the last installment[1]. This will help us to understand both phenomena better.

The first titular president of the French republic, Louis-Napoleon Bonaparte, permitted Foucault to use the Pantheon in Paris to give a demonstration of his pendulum (with a 67 meter wire and a 28 kg pendulum bob) on 31 March 1851. In this impressive experiment, one could see the plane of oscillation of the pendulum rotating in a clockwise direction (when viewed from the top) with a frequency $\omega = \Omega \cos\theta$, where Ω is the angular frequency of Earth's rotation and θ is the co-latitude of Paris. (That is, θ is the standard polar angle in spherical polar coordinates with the z-axis being the axis of rotation of Earth. So $\pi/2 - \theta$ is the geographical latitude). Foucault claimed, quite correctly, that this effect arises due to the rotation of the Earth and thus showed that one can demonstrate the rotation of the Earth by an *in situ* experiment without looking at celestial objects.

[1] Thomas Precession, *Resonance*, Vol.13, No.7, pp.610–618, July 2008.

This result is quite easy to understand if the experiment was performed at the poles or equator (instead of Paris!). The situation at the North Pole is as shown in *Figure* 1. Here we see the Earth as rotating (from west to east, in the counter-clockwise direction when viewed from the top) underneath the pendulum, making one full turn in 24 hours. It seems reasonable to deduce from this

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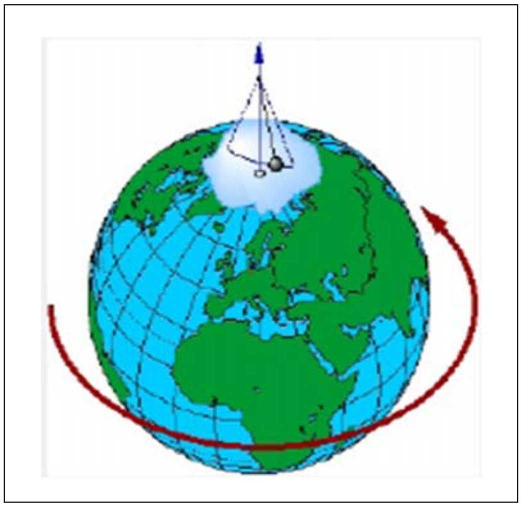


Figure 1.

that, as viewed from Earth, the plane of oscillation of the pendulum will make one full rotation in 24 hours; so the angular frequency ω of the rotation of the plane of the Foucault pendulum is just $\omega = \Omega$. (Throughout the discussion we are concerned with the rotation of the plane of oscillation of the pendulum; not the period of the pendulum $2\pi/\nu$, which – of course – is given by the standard formula involving the length of the suspension wire, etc.). At the equator, on the other hand, the plane of oscillation does not rotate. So the formula, $\omega = \Omega \cos\theta$, captures both the results correctly.

It is trivial to write down the equations of motion for the pendulum bob in the rotating frame of the Earth and solve them to obtain this result [1, 2] at the linear order in Ω. Essentially, the Foucault pendulum effect arises due to the Coriolis force in the rotating frame of the Earth which leads to an acceleration $2\mathbf{v} \times \mathbf{\Omega}$, where $\mathbf{v}$, the velocity of the pendulum bob, is directed tangential to the Earth's surface to a good approximation. If we choose a local coordinate system with the Z-axis pointing normal to the surface of the Earth and the X, Y coordinates in the tangent plane at the location, then it is easy to show that the equations of motion for the

Jean Bernard Lèon Foucault (1819 –1868) was a French physicist, famous for the demonstration of Earth's rotation with his pendulum. Although Earth's rotation was not unknown then, but this easy-to-see experiment caught everyone's imagination.

pendulum bob are well approximated by

$$\ddot{X} + \nu^2 X = 2\Omega_z \dot{Y}; \qquad \ddot{Y} + \nu^2 Y = -2\Omega_z \dot{X}, \tag{1}$$

where ν is the period of oscillation of the pendulum and $\Omega_z = \Omega \cos\theta$ is the normal component of the Earth's angular velocity. In arriving at these equations we have ignored the terms quadratic in Ω and the vertical displacement of the pendulum. The solution to this equation is obtained easily by introducing the variable $q(t) \equiv X(t) + iY(t)$. This satisfies the equation

$$\ddot{q} + 2i\Omega_z \dot{q} + \nu^2 q = 0. \tag{2}$$

The solution, to the order of accuracy we are working with, is given by

$$q = X(t) + iY(t) = (X_0(t) + iY_0(t)) \exp(-i\Omega_z t), \tag{3}$$

where $X_0(t), Y_0(t)$ is the trajectory of the pendulum in the absence of Earth's rotation. It is clear that the net effect of rotation is to cause a shift in the plane of rotation at the rate $\Omega_z = \Omega \cos\theta$. Based on this knowledge and the results for the pole and the equator one can give a 'plain English' derivation of the result for intermediate latitudes by saying something like: "Obviously, it is the component of $\mathbf{\Omega}$ normal to the Earth at the location of the pendulum which matters and hence $\omega = \Omega \cos\theta$."

The first-principle approach, based on (1), of course has the advantage of being rigorous and algorithmic; for example, if you want to take into account the effects of ellipticity of Earth, you can do that if you work with the equations of motion. But it does not give you an intuitive understanding of what is going on, and much less a unified view of all related problems having the same structure. We shall now describe an approach to this problem which has the advantage of providing a clear geometrical picture and connecting it up – somewhat quite surprisingly – with Thomas precession discussed in the last installment.

One point which causes some confusion as regards the Foucault pendulum is the following. While analyzing the behavior of the pendulum at the pole, one assumes that the plane of rotation remains fixed while the Earth rotates underneath it. If we make the same claim for a pendulum experiment done at an intermediate latitude, – i.e., if we say that the plane of oscillation remains invariant with respect to, say, the "fixed stars" and the Earth rotates underneath it – it seems natural that the period of rotation of the pendulum plane should always be 24 hours irrespective of the location! This, of course, is not true and it is also intuitively obvious that nothing happens to the plane of rotation at the equator. In this way of approaching the problem, it is not very clear how exactly the Earth's rotation influences the motion of the pendulum.

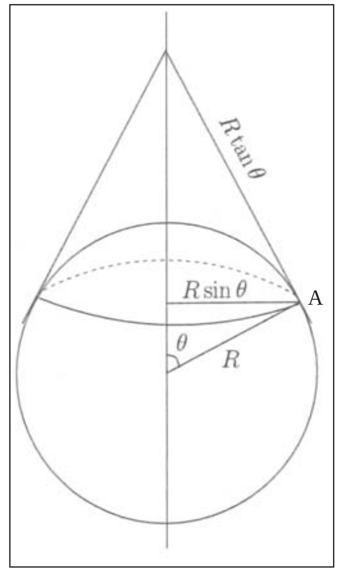


Figure 2.

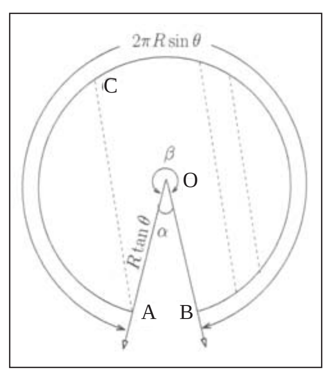
To provide a geometrical approach to this problem, we will rephrase it as follows [3, 4]. The plane of oscillation of the pendulum can be characterized by a vector normal to it or equivalently by a vector which is lying in the plane *and* tangential to the Earth's surface. Let us now introduce a cone which is coaxial with the axis of rotation of the Earth and with its surface tangential to the Earth at the latitude of the pendulum (see *Figure* 2). The base radius of such a cone will be $R\sin\theta$, where R is the radius of the Earth and the slant height of the cone will be $R\tan\theta$. Such a cone can be built out of a sector of a circle (as shown in *Figure* 3) having the circumference $2\pi R\sin\theta$ and radius $R\tan\theta$ by identifying the lines OA and OB. The 'deficit angles' of the cone, α and $\beta \equiv 2\pi - \alpha$, satisfy the relations:

$$(2\pi - \alpha)R\tan\theta = 2\pi R\sin\theta \qquad (4)$$

which gives

$$\alpha = 2\pi(1-\cos\theta); \qquad \beta = 2\pi\cos\theta. \qquad (5)$$

Figure 3.



The behavior of the plane of the Foucault pendulum

The plane of oscillation of the pendulum will rotate with respect to a coordinate system fixed on the Earth, but it will always coincide with the lines drawn on the cone which remain fixed relative to the fixed stars. (*Figures* 2,3)

can be understood very easily in terms of this cone. Initially, the Foucault pendulum is started out oscillating in some arbitrary direction at the point A, say. The direction of oscillation can be indicated by some straight line drawn along the surface of the cone (like AC in *Figure* 3). While the plane of oscillation of the pendulum will rotate with respect to a coordinate system fixed on the Earth, it will always coincide with the lines drawn on the cone which remain fixed relative to the fixed stars. When the Earth makes one rotation, we move from A to B in the flattened out cone in *Figure* 3. Physically, of course, we identify the two points A and B with the same location on the surface of the Earth. But when a vector is moved around a curve along the lines described above, on the curved surface of Earth, its orientation does not return to the original value. It is obvious from *Figure* 3 that the orientation of the plane of rotation (indicated by a vector in the plane of rotation and tangential to the Earth's surface at B) will be different from the corresponding vector at A. (This process is called parallel transport and the fact that a vector changes on parallel transport around an arbitrary closed curve on a curved surface is a well-known result in differential geometry and general relativity.)

Clearly, the orientation of the vector changes by an angle $\beta = 2\pi \cos\theta$ during one rotation of Earth with period T. Since the rate of change is uniform throughout because of the steady state nature of the problem, the angular velocity of the rotation of the pendulum plane is given by

$$\omega = \frac{\beta}{T} = \frac{2\pi}{T}\cos\theta = \Omega\cos\theta. \qquad (6)$$

This is precisely the result we were after. The key geometrical idea was to relate the rotation of the plane of the Foucault pendulum to the parallel transport of a vector characterizing the plane, around a closed curve on the surface of Earth. When this closed curve is not a geodesic – and we know that a curve of constant latitude is not a geodesic – the orientation of this vector changes when it completes one loop. There are more sophisticated ways of calculating how much the orientation changes for a given curve on a curved surface. But in the case of a sphere, the trick of an enveloping cone provides a simple procedure. (When the pendulum is located in the equator, the closed curve is the equator itself; this, being a great circle is a geodesic on the sphere and the vector does not get 'disoriented' on going around it. So the plane of the pendulum does not rotate in this case.)

This derivation also allows one to understand the Thomas precession of the spin of a particle.

This is good, but as I said, things get better. One can show that an almost identical approach allows one to determine the Thomas precession of the spin of a particle (say, an electron) moving in a circular orbit around a nucleus [5].

We saw in the last installment [6] that the rate of Thomas precession is given, in general, by an expression of the form

$$\boldsymbol{\omega}\mathrm{d}t = (\cosh\chi - 1)\,(\mathrm{d}\hat{\boldsymbol{n}} \times \hat{\boldsymbol{n}})\,, \tag{7}$$

where $\tanh\chi = v/c$ and v is the velocity of the particle. In the case of a particle moving on a circular trajectory, the magnitude of the velocity remains constant and we can integrate this expression to obtain the net angle of precession during one orbit. For a circular orbit, $\mathrm{d}\hat{\boldsymbol{n}}$ is always perpendicular to $\hat{\boldsymbol{n}}$ so that $\hat{\boldsymbol{n}} \times \mathrm{d}\hat{\boldsymbol{n}}$ is essentially $\mathrm{d}\theta$ which integrates to give a factor 2π. Hence the net angle of Thomas precession during one orbit is given by

$$\Phi = 2\pi(\cosh\chi - 1). \tag{8}$$

The similarity between the net angle of turn of the Foucault pendulum and the net Thomas precession angle is now obvious when we compare (8) with (5). We know that in the case of Lorentz transformations, one replaces

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real angles by imaginary angles which accounts for the difference between the cos and cosh factors. What we need to do is to make this analogy mathematically precise which will be our next task. It will turn out that the sphere and the cone we introduced in the real space, to study the Foucault pendulum, have to be introduced in the velocity space to analyze Thomas precession.

As a warm-up to exploring the relativistic velocity space, let us start by asking the following question: Consider two frames S_1 and S_2 which move with velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ with respect to a third inertial frame S_0. What is the magnitude of the relative velocity between the two frames? This is most easily done using Lorentz invariance and four-vectors (and to simplify notation we will use units with $c = 1$). We can associate with the 3-velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, the corresponding four-velocities, given by $u_1^i = (\gamma_1, \gamma_1\boldsymbol{v}_1)$ and $u_2^i = (\gamma_2, \gamma_2\boldsymbol{v}_2)$ with all the components being measured in S_0. On the other hand, with respect to S_1, this four-vector will have the components $u_1^i = (1, 0)$ and $u_2^i = (\gamma, \gamma\boldsymbol{v})$, where $\boldsymbol{v}$ (by definition) is the relative velocity between the frames. To determine the magnitude of this quantity, we note that in this frame S_1 we can write $\gamma = -u_{1i}u_2^i$. But since this expression is Lorentz invariant, we can evaluate it in any inertial frame. In S_0, with $u_1^i = (\gamma_1, \gamma_1\boldsymbol{v}_1), u_2^i = (\gamma_2, \gamma_2\boldsymbol{v}_2)$ this has the value

$$\gamma = (1 - v^2)^{-1/2} = \gamma_1\gamma_2 - \gamma_1\gamma_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2. \tag{9}$$

Simplifying this expression we get

$$v^2 = \frac{(1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2)^2 - (1 - v_1^2)(1 - v_2^2)}{(1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2)^2}$$

$$= \frac{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}{(1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2)^2}. \tag{10}$$

Let us next consider a 3-dimensional abstract space in which each point represents a velocity of a Lorentz frame measured with respect to some fiducial frame. We are interested in defining the notion of 'distance' between two points in this velocity space. Consider two nearby points which correspond to velocities $\boldsymbol{v}$ and $\boldsymbol{v} + \mathrm{d}\boldsymbol{v}$ that differ by an infinitesimal quantity. By analogy with the usual 3-dimensional flat space, one would have assumed that the 'distance' between these two points is just

$$|\mathrm{d}\boldsymbol{v}|^2 = \mathrm{d}v_x^2 + \mathrm{d}v_y^2 + \mathrm{d}v_z^2 = \mathrm{d}v^2 + v^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{11}$$

where $v = |\boldsymbol{v}|$ and (θ, ϕ) denote the direction of $\boldsymbol{v}$. In non-relativistic physics, this distance also corresponds to the magnitude of the relative velocity between the two frames. However, we have just seen that the relative velocity between two frames in relativistic mechanics is different and given by (10). It is more natural to define the distance between the two points in the velocity space to be the relative velocity between the respective frames. In that case, the infinitesimal 'distance' between the two points in the velocity space will be given by (10) with $\boldsymbol{v}_1 = \boldsymbol{v}$ and $\boldsymbol{v}_2 = \boldsymbol{v} + \mathrm{d}\boldsymbol{v}$. So

$$\mathrm{d}l_v^2 = \frac{(\mathrm{d}\boldsymbol{v})^2 - (\boldsymbol{v} \times \mathrm{d}\boldsymbol{v})^2}{(1 - v^2)^2}. \tag{12}$$

Using the relations

$$(\boldsymbol{v} \times \mathrm{d}\boldsymbol{v})^2 = v^2(\mathrm{d}\boldsymbol{v})^2 - (\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v})^2; \quad (\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v})^2 = v^2(\mathrm{d}v)^2 \tag{13}$$

and using (11) where θ, ϕ are the polar and azimuthal angles of the direction of $\boldsymbol{v}$, we get

$$\mathrm{d}l_v^2 = \frac{\mathrm{d}v^2}{(1 - v^2)^2} + \frac{v^2}{1 - v^2}(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2). \tag{14}$$

If we use the rapidity χ in place of v through the equation $v = \tanh\chi$, the line element in (14) becomes:

$$\mathrm{d}l_v^2 = \mathrm{d}\chi^2 + \sinh^2\chi(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2). \tag{15}$$

This is an example of a *curved* space within the context of special relativity. This particular space is called (three-dimensional) Lobachevsky space.

If we now change from real angles to the imaginary ones, by writing $\chi = i\eta$, the line element becomes

$$-\mathrm{d}l_v^2 = \mathrm{d}\eta^2 + \sin^2\eta(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \qquad (16)$$

which (except for an overall sign which is irrelevant) represents the distances on a 3-sphere having the three angles η, θ and ϕ as its coordinates.

Of these three angles, θ and ϕ denote the direction of velocity in the real space as well. When a particle moves in the $x - y$ plane in the real space, its velocity vector lies in the $\theta = \pi/2$ plane and the relevant part of the metric reduces to

$$\mathrm{d}L_v^2 = \mathrm{d}\eta^2 + \sin^2\eta\mathrm{d}\phi^2 \qquad (17)$$

which is just a metric on the 2-sphere. Further, if the particle is moving on a circular orbit with constant magnitude for the velocity, then it follows a curve of $\eta =$ constant on this 2-sphere. The analogy with the Foucault pendulum, which moves on a constant latitude curve, is now complete. If the particle carries a spin, the orbit will transport the spin vector along this circular orbit. As we have seen earlier, the orientation of the vector will not coincide with the original one when the orbit is completed and we expect a difference of $2\pi(1 - \cos\eta) = 2\pi(1 - \cosh\chi)$. So the magnitude of the Thomas precession, over one period is given precisely by (8). I will let you work out the details exactly in analogy with the Foucault pendulum and convince yourself that they have the same geometrical interpretation.

When one moves along a curve in the velocity space, one is sampling different (instantaneously) co-moving Lorentz frames obtained by Lorentz boosts along different directions. As we described in the last installment, Lorentz boosts along different directions do not, in general, commute. This leads to the result that if we move along a closed curve in the velocity space (treated as representing different Lorentz boosts) the orientation of the spatial axes would have changed when we complete the loop.

It turns out that the ideas described above are actually of far more general validity. Whenever a vector is transported around a closed curve on the surface of a sphere, the net change in its orientation can be related to the solid angle subtended by the area enclosed by the curve. In the case of the Foucault pendulum, the relevant vector describes the orientation of the plane of the pendulum and the transport is around a circle on the surface of the Earth. In the case of Thomas precession, the relevant vector is the spin of the particle and the transport occurs in the velocity space. Ultimately, both the effects – the Foucault pendulum and Thomas precession – arise because the space in which one is parallel transporting the vector (surface of Earth, relativistic velocity space) is curved.

Suggested Reading

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