Snippets of Physics

7. Thomas Precession

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Thomas precession is a curious effect in special relativity which is purely kinematical in origin and illustrates some important features of the Lorentz transformation. It also has a beautiful geometric interpretation. We will explore these in this and the next installment.

The simplest context in which the Thomas precession arises is when an object with an intrinsic spin (like an electron or a gyroscope) moves in a closed orbit with variable velocity – an example being the electron orbiting the nucleus in an atom, treated along classical lines. It turns out that, due to Thomas precession, the effective energy of coupling between the spin and the orbital angular momentum of the electron picks up an extra factor of (1/2) which, of course, has experimentally verifiable consequences. Naively, you might have thought that any special relativistic effect should lead to a correction which is of the order of $(v/c)^2$ and hence will be a very weak effect for an electron in an atom. This is indeed true. But experimentally observable effects of the spin-orbit interaction are *also* relativistic effects. These arise because, in the instantaneous rest frame of the orbiting electron, the Coulomb field (Ze^2/r^2) of the nucleus gives rise to a magnetic field $(v/c)(Ze^2/r^2)$. This magnetic field couples to the magnetic moment $(e\hbar/2m_ec)$ of the electron. So any other effect which is of the order of $\mathcal{O}(v^2/c^2)$ will change the observable consequences by order unity factors.

It turns out that this precession also has an interesting geometrical interpretation that allows one to relate it to other – apparently unconnected – physical phenomena

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like the rotation of the plane of the Foucault pendulum. In this installment I will provide a straightforward (and possibly not very inspiring) derivation of the Thomas precession. Next month we will explore the Foucault pendulum and the geometrical relationship.

Consider the standard Lorentz transformation equations between two inertial frames which are in relative motion along the x-axis with a speed $v \equiv c\beta$. This is given by $x = \gamma(x' + vt'), t = \gamma(t' + vx'/c^2)$, where $\gamma = (1-\beta^2)^{-1/2}$. We know that the Lorentz transformation leaves the quantity $s^2 \equiv (-c^2t^2 + |\boldsymbol{x}|^2)$ invariant. A quadratic expression of this form is similar to the length of a vector in three dimensions which is invariant under rotation of the coordinate axes. This suggests that the transformation between the inertial frames can be thought of as a rotation in four-dimensional space. The rotation must be in the t–x plane characterized by a parameter, say, ψ. Indeed, the Lorentz transformation can be equivalently written as

$$x = x' \cosh\psi + ct' \sinh\psi, \qquad ct = x' \sinh\psi + ct' \cosh\psi. \tag{1}$$

with $\tanh\psi = (V/c)$, which determines the parameter ψ (called the *rapidity*) in terms of the relative velocity between the two frames. Equation (1) can be thought of as a rotation by a complex angle $i\psi$.

Two successive Lorentz transformations with velocities v_1 and v_2, *along the same direction* x, will correspond to two successive rotations in the t–x plane by angles, say, ψ_1 and ψ_2. Since two rotations in the same plane about the same origin commute, it is obvious that these two Lorentz transformations commute and are equivalent to a rotation by an angle $\psi_1 + \psi_2$ in the t–x plane. This results in a single Lorentz transformation with a velocity parameter given by the relativistic sum of the two velocities v_1 and v_2. Note that the rapidities simply add while the velocity addition formula is more complicated.

The transformation between inertial frames can be thought of as a rotation in four-dimensional space.

The situation, however, changes in the case of Lorentz transformations along two different directions. These will correspond to rotations in two different planes and it is well known that such rotations will not commute. The order in which the Lorentz transformations are carried out is important if they are along different directions. Suppose a frame S_1 is moving with a velocity $\boldsymbol{v}_1 = v_1\boldsymbol{n}_1$ (where $\boldsymbol{n}_1$ is a unit vector) with respect to a reference frame S_0 and we do a Lorentz boost to connect the coordinates of these two frames. Now suppose we do another Lorentz boost with a velocity $\boldsymbol{v}_2 = v_2\boldsymbol{n}_2$ to go from S_1 to S_2. We want to know what kind of transformation will now take us directly from S_0 to S_2. If $\boldsymbol{n}_1 = \boldsymbol{n}_2$, then the two Lorentz transformation are along the same axis and one can go from S_0 to S_2 by a single Lorentz transformation. But if the two directions $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are different, then this is not possible. It turns out that in addition to the Lorentz transformation one also has to rotate the spatial coordinates by a particular amount.

This is the root cause of Thomas precession. When a body is moving in an accelerated trajectory with the direction of velocity vector changing continuously, the instantaneous Lorentz frames are obtained by boosts along different directions at each instant. Since such successive boosts are equivalent to a boost plus a rotation of spatial axes, there is an effective rotation of the coordinate axes which occurs in the process. If the body carries an intrinsic vector (like spin) with it, the orientation of that vector will undergo a shift.

The root cause of Thomas precession is this: when a body is accelerated, with its velocity vector changing continuously, the instantaneous Lorentz frames are obtained by boosts along different directions, and there is an effective rotation of coordinate axes which occurs in the process.

After all that English, let us do some maths to establish the idea rigorously. To do this we need the Lorentz transformations connecting two different frames of references, when one of them is moving along an arbitrary direction $\boldsymbol{n}$ with speed $V \equiv \beta c$. The time coordinates are related by the obvious formula

$$x'^0 = \gamma(x^0 - \boldsymbol{\beta} \cdot \boldsymbol{x}), \tag{2}$$

where we are using the notation $x^i = (x^0, \boldsymbol{x}) = (ct, \boldsymbol{x})$ to denote the four-vector coordinates. To obtain the transformation of the spatial coordinate, we first write the spatial vector $\boldsymbol{x}$ as a sum of two vectors; $\boldsymbol{x}_\parallel = \mathbf{V}(\mathbf{V}\cdot\boldsymbol{x})/V^2$ which is parallel to the velocity vector and $\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel$ which is perpendicular to the velocity vector. We know that, under the Lorentz transformation, we have $\boldsymbol{x}'_\perp = \boldsymbol{x}_\perp$ while $\boldsymbol{x}'_\parallel = \gamma(\boldsymbol{x}_\parallel - \mathbf{V}t)$. Expressing everything again in terms of $\boldsymbol{x}$ and $\boldsymbol{x}'$, it is easy to show that the final result can be written in the vectorial form (with $\boldsymbol{\beta} = \beta\boldsymbol{n}$) as:

$$\boldsymbol{x}' = \boldsymbol{x} + \frac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x^0. \tag{3}$$

Equations (2) and (3) give the Lorentz transformation between two frames moving along an arbitrary direction.

We want to use this result to determine the effect of two consecutive Lorentz transformations for the case in which both $\boldsymbol{v}_1 = v_1\boldsymbol{n}_1$ and $\boldsymbol{v}_2 = v_2\boldsymbol{n}_2$ are small in the sense that $v_1 \ll c$, $v_2 \ll c$. Let the first Lorentz transformation take the four vector $x^b = (ct, \mathbf{x})$ to x_1^b and the second Lorentz transformation take this further to x_{21}^a. Performing the same two Lorentz transformations in reverse order leads to the vector which we will denote by x_{12}^a. We are interested in the difference $\delta x^a \equiv x_{21}^a - x_{12}^a$ to the lowest nontrivial order in (v/c). Since this involves product of two Lorentz transformations, we need to compute it keeping all terms up to *quadratic* order in v_1 and v_2. Explicit computation, using, (3) and (2) now gives (try it out!)

$$\begin{aligned}
x_{21}^0 &\approx [1 + \frac{1}{2}(\boldsymbol{\beta}_2 + \boldsymbol{\beta}_1)^2]x^0 - (\boldsymbol{\beta}_2 + \boldsymbol{\beta}_1)\cdot\boldsymbol{x}, \\
\boldsymbol{x}_{21} &\approx \boldsymbol{x} - (\boldsymbol{\beta}_2 + \boldsymbol{\beta}_1)x^0 + [\boldsymbol{\beta}_2(\boldsymbol{\beta}_2\cdot\boldsymbol{x}) + \boldsymbol{\beta}_1(\boldsymbol{\beta}_1\cdot\boldsymbol{x})] + \\
&\quad \boldsymbol{\beta}_2(\boldsymbol{\beta}_1\cdot\boldsymbol{x}),
\end{aligned} \tag{4}$$

accurate to $\mathcal{O}(\beta^2)$. It is obvious that the terms which are symmetric under the exchange of 1 and 2 in the above expression will cancel out when we compute $\delta x^a \equiv x^a_{21} - x^a_{12}$. Hence, we immediately get $\delta x^0 = 0$ to this order of accuracy. In the spatial components the only term which survives is the one arising from last term in the expression for $\boldsymbol{x}_{21}$ which gives

$$\delta \boldsymbol{x} = [\boldsymbol{\beta}_2(\boldsymbol{\beta}_1 \cdot \boldsymbol{x}) - \boldsymbol{\beta}_1(\boldsymbol{\beta}_2 \cdot \boldsymbol{x})] = \frac{1}{c^2}(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \times \boldsymbol{x}. \quad (5)$$

Comparing this with the standard result for infinitesimal rotation of coordinates $\delta \boldsymbol{x} = \boldsymbol{\Omega} \times \boldsymbol{x}$, we find that the net effect of two Lorentz transformations leaves a residual *spatial rotation* about the direction $\boldsymbol{v}_1 \times \boldsymbol{v}_2$. Since this result was obtained by taking the difference between two successive Lorentz transformations, $\delta \boldsymbol{x} \equiv \boldsymbol{x}_{21} - \boldsymbol{x}_{12}$, we can think of each one contributing an effective rotation by the amount $(1/2)(\boldsymbol{v}_1 \times \boldsymbol{v}_2)/c^2$. Consider now a particle with a spin moving a circular orbit. (For example, it could be an electron in an atom; the classical analysis continues to apply mainly because the effect is purely kinematic!). At two instances in time t and $t+\delta t$, the velocity of the electron will be in different directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_1 + \boldsymbol{a}\delta t$, where $\boldsymbol{a}$ is the acceleration. This should lead to a change in the angle of orientation of the axes by the amount

$$\delta \boldsymbol{\Omega} = \frac{1}{2}\frac{(\boldsymbol{v}_1 \times \boldsymbol{v}_2)}{c^2} = \frac{1}{2}\frac{(\boldsymbol{v}_1 \times \boldsymbol{a})}{c^2}\ \delta t \quad (6)$$

corresponding to the angular velocity $\boldsymbol{\omega} = \delta\boldsymbol{\Omega}/\delta t = (1/2)(\boldsymbol{v}_1 \times \boldsymbol{a})/c^2$. This is indeed the correct expression for the Thomas precession in the nonrelativistic limit (since we had assumed $v_1 \ll c, v_2 \ll c$).

Let me now outline a rigorous derivation of this effect, leaving the algebraic details for you to figure out! To set the stage, we again begin with the rotations in 3-dimensional space. A given rotation can be defined by specifying the unit vector $\boldsymbol{n}$ in the direction of the axis of rotation and the angle θ through which the axes are rotated. We associate with this rotation a 2×2 matrix

$$R(\theta) = \cos(\theta/2) - i(\boldsymbol{\sigma} \cdot \boldsymbol{n}) \sin(\theta/2) = \exp -\frac{i\theta}{2}(\boldsymbol{\sigma} \cdot \boldsymbol{n}), \tag{7}$$

where σ_α are the standard Pauli matrices and the $\cos(\theta/2)$ term is considered to be multiplied by the unit matrix though it is not explicitly indicated. The equivalence of the two forms – the exponential and trigonometric – of $R(\theta)$ in (7) can be demonstrated by expanding the exponential in a power series and using the easily proved relation $(\boldsymbol{\sigma} \cdot \boldsymbol{n})^2 = 1$. (Incidentally, the occurrence of the angle $\theta/2$ has a simple geometrical origin: A rotation through an angle θ about a given axis may be visualized as the consequence of successive reflections in two planes which meet along the axis at an angle $\theta/2$.) We can also associate with a 3-vector $\boldsymbol{x}$, the 2×2 matrix $X = \boldsymbol{x} \cdot \boldsymbol{\sigma}$. The effect of any rotation can now be concisely described by the matrix relation $X' = RXR^*$.

Since we can think of Lorentz transformations as rotations by an imaginary angle, all these results generalize, in a natural fashion, to Lorentz transformations. We shall associate with a Lorentz transformation in the direction $\boldsymbol{n}$ with the speed $v = c \tanh \alpha$, the 2×2 matrix

$$L = \cosh(\alpha/2) + (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \sinh(\alpha/2) = \exp \frac{1}{2}(\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}). \tag{8}$$

The change from trigonometric functions to hyperbolic functions is in accordance with the fact that Lorentz transformations correspond to rotation by an *imaginary* angle. Just as in the case of rotations, we can associate to any event $x^i = (x^0, \boldsymbol{x})$, a (2×2) matrix $P \equiv x^i \sigma_i$ where σ_0 is the identity matrix and σ_α are the Pauli matrices. Under a Lorentz transformation along the direction $\hat{\boldsymbol{n}}$ with speed v, the event x^i goes to $x^{i'}$ and P goes P'. (By convention σ_i's do not change.) They are

The change from trigonometric functions to hyperbolic functions is in accordance with the fact that Lorentz transformations correspond to rotation by an *imaginary* angle.

with

$$P' = LPL^* \tag{9}$$

where L is given by (8).

Consider a frame S_0 which is an inertial, laboratory frame and let $S(t)$ be a Lorentz frame comoving with a particle (with a spin) at time t. These two frames are related to each other by a Lorentz transformation with a velocity $\boldsymbol{v}$. Consider a pure Lorentz boost in the *comoving* frame of the particle which changes its velocity relative to the lab frame from $\boldsymbol{v}$ to $\boldsymbol{v} + \mathrm{d}\boldsymbol{v}$. We know that the resulting final configuration cannot be reached from S_0 by a pure boost and we require a rotation by some angle $\delta\boldsymbol{\theta} = \boldsymbol{\omega}\mathrm{d}t$ followed by a simple boost. This leads to the relation, in terms of the 2×2 matrices corresponding to the rotation and Lorentz transformations, as:

$$L(\boldsymbol{v} + \mathrm{d}\boldsymbol{v})R(\boldsymbol{\omega}\mathrm{d}t) = L_{\mathrm{comov}}(\mathrm{d}\boldsymbol{v})L(\boldsymbol{v}). \tag{10}$$

The right-hand side represents, in matrix form, two Lorentz transformations. The left-hand side represents the same effect in terms of one Lorentz transformation and one rotation – the parameters of which are at present unknown. In the right-hand side of (10) the matrix $L_{\mathrm{comov}}(\mathrm{d}\boldsymbol{v})$ has a subscript 'comoving' to stress the fact that this operation corresponds to a pure boost *only* in the comoving frame and *not* in the lab frame. To take care of this, we do the following: We first bring the particle to rest by applying the inverse Lorentz transformation operator $L^{-1}(\boldsymbol{v}) = L(-\boldsymbol{v})$. Then we apply a boost $L(\boldsymbol{a}_{\mathrm{comov}}\mathrm{d}\tau)$, where $\boldsymbol{a}_{\mathrm{comov}}$ is the acceleration of the system in the comoving frame. Since the object was at rest initially, *this operation* can be characterized by a pure boost. Finally, we transform back from the lab to the moving frame by applying $L(\boldsymbol{v})$. Therefore we have the relation

$$L_{\mathrm{comov}}(\mathrm{d}\boldsymbol{v}) = L(\boldsymbol{v})L(\boldsymbol{a}_{\mathrm{comov}}\mathrm{d}\tau)L(-\boldsymbol{v}). \tag{11}$$

Using this is in (10), we get

$$L(\boldsymbol{v} + \mathrm{d}\boldsymbol{v})R(\boldsymbol{\omega}\mathrm{d}t) = L(\boldsymbol{v})L(\boldsymbol{a}_{\mathrm{comov}}\mathrm{d}\tau).$$

In this equation, the unknowns are $\boldsymbol{\omega}$ and $\boldsymbol{a}_{\mathrm{comov}}$. Moving the unknown terms to the left-hand side, we have the equation,

$$R(\boldsymbol{\omega}\mathrm{d}t)L(-\boldsymbol{a}_{\mathrm{comov}}\mathrm{d}\tau) = L(-[\boldsymbol{v} + \mathrm{d}\boldsymbol{v}])L(\boldsymbol{v}), \qquad (12)$$

which can be solved for $\boldsymbol{\omega}$ and $\boldsymbol{a}_{\mathrm{comov}}$. If we denote the rapidity parameters for the two infinitesimally separated Lorentz boosts by α and $\alpha' \equiv \alpha + \mathrm{d}\alpha$ and the corresponding directions by $\boldsymbol{n}$ and $\boldsymbol{n}' \equiv \boldsymbol{n} + \mathrm{d}\boldsymbol{n}$, then this matrix equation can be expanded to first order quantities to give

$$\begin{aligned}
&1 - (i\boldsymbol{\omega}\mathrm{d}t + \boldsymbol{a}\mathrm{d}\tau)\cdot\frac{\boldsymbol{\sigma}}{2} \\
&= [\cosh(\alpha'/2) - (\boldsymbol{n}'\cdot\boldsymbol{\sigma})\sinh(\alpha'/2)] \\
&[\cosh(\alpha/2) - (\boldsymbol{n}\cdot\boldsymbol{\sigma})\sinh(\alpha/2)]. \qquad (13)
\end{aligned}$$

Performing the necessary Taylor series expansion in $\mathrm{d}\alpha$ and $\mathrm{d}\boldsymbol{n}$ in the right-hand side and identifying the corresponding terms on both sides, we find – after some algebra! – that $\boldsymbol{a}_{\mathrm{comov}} = \hat{\boldsymbol{n}}(\mathrm{d}\alpha/\mathrm{d}\tau) + (\sinh\alpha)(\mathrm{d}\hat{\boldsymbol{n}}/\mathrm{d}\tau)$ and more importantly,

$$\boldsymbol{\omega} = (\cosh\alpha - 1)\left(\frac{\mathrm{d}\hat{\boldsymbol{n}}}{\mathrm{d}t} \times \hat{\boldsymbol{n}}\right) \qquad (14)$$

with $\tanh\alpha = (v/c)$. (This result for $\boldsymbol{\omega}\mathrm{d}t$ has a nice geometrical interpretation which we will discuss next month.) Expressing everything in terms of the velocity, it is easy to show that the expression for $\boldsymbol{\omega}$ is equivalent to

$$\omega = \frac{\gamma^2}{\gamma + 1}\frac{\boldsymbol{a}\times\boldsymbol{v}}{c^2} = (\gamma - 1)\frac{(\boldsymbol{v}\times\boldsymbol{a})}{v^2}. \qquad (15)$$

In the nonrelativistic limit ($\boldsymbol{v} << c$), this gives a precessional angular velocity $\boldsymbol{\omega} \cong (1/2c^2)(\boldsymbol{a}\times\boldsymbol{v})$ which the

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spin will undergo because of the non-commutativity of Lorentz transformations in different directions. Working out the details of the derivation given above is a worthwhile exercise in special relativity.

Suggested Reading

[1] C W Misner, K S Thorne and J A Wheeler, *Gravitation*, Chapter 41, (Freeman), 1973.

Information and Announcements

Refresher Course on RS and GIS Applications for Water and Environmental Technologies

The Centre for Water Resources, Institute of Science and Technology, JNTU, Hyderabad

25th August 2008 to 16th September 2008

The Centre for Water Resources, Institute of Science and Technology, JNTU, Hyderabad, Andhra Pradesh is organizing an UGC sponsored above refresher course for faculty members working in universities and colleges. This course is designed to help the participants upgrade their academic and research activities. The course will be further supplemented by field trips, hands on practical training on Arc GIS 9.1 and ERDAS 8.7 and interactions with industry/academia experts. Eligible candidates would be provided to and fro second class railway fare by the shortest route for attending the programme and each participant shall pay an amount of Rs.500/- towards registration fee. Free boarding and lodging will be provided for outstation participants. This course is planned as a residential programme and stay in the University Guest House at Kukatpally Campus of JNTU, Hyderabad, is compulsory for outstation participants. Selection will be based on a first-come-first-served basis. Application form can be downloaded from the Centre for Water Resources web site www.cwr.co.in and also from the JNT University website www.jntu.ac.in. Last date for registration is 18th August, 2008.

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