Snippets of Physics

6. The Logarithms of Physics

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Scaling arguments and dimensional analysis are powerful tools in physics which help you to solve several interesting problems. And when the scaling arguments fail, as in the examples discussed here, we are led to a more fascinating situation.

Let us begin this time by revisiting a problem which is beaten to death in standard textbooks in electrodynamics – except that we will do it in a slightly different manner and get ourselves all tied up in knots. Consider an infinite straight line charge located along the $y-$axis with the charge density per unit length being λ. We are interested in determining the electric field everywhere due to this line charge.

The standard solution to this problem is ridiculously simple. You first argue, based on the symmetry, that the electric field at any given point is in the $x - z$ plane and depends only on the distance from the line charge. So we can arrange the coordinate system such that the point at which we want to calculate the field is at $(x, 0, 0)$. If we now enclose the line charge by an imaginary concentric cylindrical surface of radius x and length L, the outward flux of electric field through the surface is $2\pi xLE$ which should be equal to 4π times the charge enclosed by the cylinder, which is $4\pi L\lambda$. This immediately gives $E = (2\lambda/x)$. [You would have noticed to your surprise that I am using the cgs units; the SI people should replace 4π by $(1/\epsilon_0)$.] Dimensionally, electric field is charge divided by square of the length and since λ is charge per unit length, everything is fine.

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We will now do it differently and in – what should be – an equivalent way. We want to compute the electrostatic potential ϕ at $(x, 0, 0)$ due to the line charge along the $y-$axis and obtain the electric field by differentiating. Obviously, the potential $\phi(x)$ can only depend on x and λ and must have the dimension of charge per unit length. If we take $\phi \sim \lambda^n x^m$, dimensional analysis immediately gives $n = 1$ and $m = 0$, so that $\phi(x) \propto \lambda$ and is independent of x! The potential is a constant and the electric field vanishes! We are in trouble.

An explicit computation of the potential from first principles makes matters worse. An infinitesimal amount of charge $\mathrm{d}q = \lambda \mathrm{d}y$ located between y and $y + \mathrm{d}y$ will lead to an electrostatic potential $\mathrm{d}q/r$ at the field point, where $r = (x^2 + y^2)^{1/2}$. So the total potential is given by

$$\phi(x) = \lambda \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{\sqrt{x^2 + y^2}} = 2\lambda \int_0^{+\infty} \frac{\mathrm{d}y}{\sqrt{x^2 + y^2}}. \quad (1)$$

Changing variables from y to $u = y/x$, the integral becomes

$$\phi(x) = 2\lambda \int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{1 + u^2}}. \quad (2)$$

This result is clearly independent of x and hence a constant which is what dimensional analysis told us. Much worse, it is an *infinite* constant since the integral diverges at the upper limit. What is going on in such a simple, classic, textbook problem?

As a first attempt in getting a sensible result, let us cut-off the integral at some length scale $y = \Lambda$. (You may think of the infinite line charge as the limit of a line charge of length 2Λ with $\Lambda \gg x$.) Using the substitution $y = x \sinh\theta$ and taking the limit $\Lambda \gg x$, we get

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$$\phi(x) = 2\lambda \int_0^\Lambda \frac{\mathrm{d}y}{\sqrt{x^2+y^2}} = 2\lambda \sinh^{-1}\left(\frac{\Lambda}{x}\right) \approx -2\lambda \ln\left(\frac{x}{2\Lambda}\right), \tag{3}$$

where we have used $\Lambda \gg x$ in arriving at the final equality. This potential does diverge when $\Lambda \to \infty$. But note that the physically observable quantity, the electric field $\mathbf{E} = -\nabla\phi$ is independent of the cut-off parameter Λ and is correctly given by $E_x = 2\lambda/x$. By introducing a cut-off, we seem to have saved the situation.

It is now clear what is going on. As the title of this article implies, the problem has to do with logarithms which allow a dimensionless function like $\ln(x/2\Lambda)$ to occur in the electrostatic potential without the electric field depending on the arbitrary scale Λ. This requires additivity on the Λ dependence; that is we need a function $f(x/\Lambda)$ which will reduce to $f(x) + f(\Lambda)$. Clearly only a logarithm will do. Once we know what is happening, it is easy to figure out other ways of getting a sensible answer. One can, for example, obtain this result from a more straightforward scaling argument by concentrating on the potential *difference* $\phi(x) - \phi(a)$, where a is some arbitrary scaling distance we introduce into the problem. From dimensional analysis, it follows that the potential difference must have the form $\phi(x) - \phi(a) = \lambda F(x/a)$, where F is a dimensionless function. Evaluating this expression for $a = 1$, say, in some units we get $\lambda F(x) = \phi(x) - \phi(1)$. Substituting back, we have the relation $\phi(x) - \phi(a) = \phi(x/a) - \phi(1)$. This functional equation has the unique solutions $\phi(x) = A \ln x + \phi(1)$. Dimensional analysis again tells you that $A \propto \lambda$. But, of course, scaling arguments cannot determine the proportionality constant. However, one can compute the potential difference by the explicit integral

$$\phi(x) - \phi(a) = 2\lambda \int_0^\infty \mathrm{d}y \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{a^2+y^2}} \right). \tag{4}$$

It is easy to see that this integral is finite. You can work it out by fairly straightforward procedures and obtain the result

$$\phi(x) - \phi(a) = -2\lambda \ln(x/a). \tag{5}$$

The numerical value of $\phi(x)$ in this expression is independent of the length scale a introduced in the problem. In that sense the scale of ϕ is determined only by λ which, as we said before, has the correct dimensions. But to ensure finite values for the expressions, we need to introduce an arbitrary length scale a which is the key feature I want to emphasize in this discussion. It turns out that such phenomena, in which naive scaling arguments breakdown due to the occurrence of logarithmic function, is a very general feature in several areas of physics especially in the study of renormalization group in high energy physics. What we have here is a very elementary manifestation of this result. In all these cases we need to smuggle into the problem a length scale to make some unobservable quantities (like the potential) finite but arrange matters such that observable quantities remain independent of this scale which we bring in.

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If you thought this was too simple, here is a more sophisticated occurrence of a logarithm for essentially the same reason.

Consider the Schrödinger equation in *two* dimensions for an attractive Dirac delta function potential $V(\mathbf{x}) = -V_0\delta(\mathbf{x})$ with $V_0 > 0$. The vector $\mathbf{x}$ is in two dimensional space and we look for a stationary bound state wavefunction $\psi(\mathbf{x})$ which satisfies the equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - V_0\delta(\mathbf{x})\right)\psi(\mathbf{x}) = -|E|\psi(\mathbf{x}), \qquad (6)$$

where $-|E|$ is the negative bound state energy. Rescaling the variables by introducing $\lambda = 2mV_0/\hbar^2$ and $\mathcal{E} = 2m|E|/\hbar^2$, this equation reduces to

$$\left(\nabla^2 + \lambda\delta(\mathbf{x})\right)\psi(\mathbf{x}) = \mathcal{E}\psi(\mathbf{x}). \qquad (7)$$

We could have done everything up to this point in any spatial dimension. In Ddimension, the Dirac delta function $\delta(\mathbf{x})$ has the dimension L^{-D}. The kinetic energy operator ∇^2, on the other hand, always has the dimension L^{-2}. This leads to a peculiar behaviour when $D = 2$. We find that, in this case, λ is dimensionless while $\mathcal{E}$ has the dimension of L^{-2}. Since the scaled binding energy $\mathcal{E}$ has to be determined entirely in terms of the parameter λ, we have a problem in our hands. There is no way we can determine the form of $\mathcal{E}$ without a dimensional constant – which we do not have.

To see the manifestation of this problem more clearly, let us solve (7). This is fairly easy to do by Fourier transforming both sides and introducing the momentum space wavefunction $\phi(\mathbf{k})$ by

$$\phi(\mathbf{k}) = \int \mathrm{d}^2\mathbf{x}\,\psi(\mathbf{x})\exp(-i\mathbf{k}\cdot\mathbf{x}). \qquad (8)$$

The left-hand side of leads to the term $[-\mathbf{k}^2\phi(\mathbf{k})+\lambda\psi(0)]$, while the right-hand side gives $\mathcal{E}\phi(\mathbf{k})$. Equating the two we get

$$\phi(\mathbf{k}) = \frac{\lambda\psi(0)}{k^2+\mathcal{E}}. \qquad (9)$$

We now integrate this equation over all $\mathbf{k}$. The left-hand side will then give $(2\pi)^2\psi(0)$ which can be cancelled out on both sides by assuming $\psi(0) \neq 0$. (This is, of course, needed for $\phi(\mathbf{k})$ in (9) to be nonzero and hence is not an additional assumption.) We then get the result

$$\frac{1}{\lambda} = \frac{1}{4\pi^2}\int \frac{\mathrm{d}^2\mathbf{k}}{k^2+\mathcal{E}} = \frac{1}{4\pi^2}\int \frac{\mathrm{d}^2\mathbf{s}}{s^2+1}. \qquad (10)$$

The Dirac delta function, in spite of the nomenclature, is strictly not a function but what mathematicians call a distribution.

The second equality is obtained by changing the integration variable to $\mathbf{s} = \mathbf{k}/\sqrt{\mathcal{E}}$. This equation is supposed to determine the binding energy $\mathcal{E}$ in terms of the parameter in the problem λ but the last expression shows that the right hand side is independent of $\mathcal{E}$! This is similar to the situation in the electrostatic problem in which we got the integral which was independent of x. In fact, just as in the electrostatic case, the integral on the right hand side diverges, confirming our suspicion. Of course, we already know that determining $\mathcal{E}$ in terms of λ is impossible due to dimensional mismatch.

One can, at this stage, take the point of view that the problem is simply ill-defined and one would be quite correct. The Dirac delta function, in spite of the nomenclature, is strictly not a function but what mathematicians call a distribution. It is defined as a limit of a sequence of functions. For example, suppose we consider a sequence of potentials

$$V(\mathbf{x}) = -\frac{V_0}{2\pi\sigma^2}\exp\left[-\frac{|\mathbf{x}|^2}{2\sigma^2}\right], \qquad (11)$$

where $\mathbf{x}$ is a 2-D vector and σ is a parameter with the dimension of length. In this case, we will again get (7) but with the Dirac delta function replaced by the Gaussian in (11). But now we have a parameter σ with the dimension of length and one can imagine the binding energy being constructed out of this. When we take the limit $\sigma \to 0$, the potential in (11) reduces to one proportional to the Dirac delta function. (This is what we meant by saying the delta function is defined as a limiting case of sequence of functions. Here the functions are Gaussians

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in (11) parametrized by σ. When we take the limit of $\sigma \to 0$ the function reduces to delta function.) The trouble is that, when we let σ go to zero, we lose the length scale in the problem and we do not know how to fix the binding energy. Of course, no one assured you that if you solve a differential equation with an input function $V(\mathbf{x};\sigma)$ which depends on a parameter σ and take a (somewhat dubious) limit of $\sigma \to 0$, then the solutions will also have a sensible limit. So one can say that the problem is ill-defined.

Rather than leaving it at that, we want to attempt here something similar to what we did in the electrostatic case. Let us evaluate the integral with a cut-off at some value $k_{\max} = \Lambda$ with $\Lambda^2 \gg \mathcal{E}$. Then we get

$$\frac{1}{\lambda} = -\frac{1}{4\pi}\ln\left(\frac{\mathcal{E}}{\Lambda^2}\right), \qquad (12)$$

which can be inverted to give the binding energy to be:

$$\mathcal{E} = \Lambda^2 \exp(-4\pi/\lambda), \qquad (13)$$

where the scale is fixed by the cut-off parameter. Of course this is what we would have got if we actually used a potential with a length scale.

There is a way of interpreting this result taking a cue from what is done in quantum field theory. The essential idea is to accept up front that the theory requires an extra scale with proper dimensions for its interpretation and treat the coupling constant as a function of the scale at which we probe the system. Having done that we arrange matters so that the observed results are actually independent of the scale we have introduced. In this case, we will define a physical coupling constant by

$$\lambda_{phy}^{-1}(\mu) = \lambda^{-1} - \frac{1}{4\pi}\ln(\Lambda^2/\mu^2) = -\frac{1}{4\pi}\ln\left(\frac{\mathcal{E}}{\mu^2}\right), \quad (14)$$

where μ is an arbitrary but finite scale. Obviously $\lambda_{phy}(\mu)$ is independent of the cut-off parameter Λ. The binding energy is now given by

$$\mathcal{E} = \mu^2 \exp(-4\pi/\lambda_{phy}(\mu)) \tag{15}$$

which, in spite of appearance, is independent of the scale μ. This is similar to our equation (5) in the electrostatic problem, in which we introduced a scale a but $\phi(x)$ was independent of a.

The breaking down of naive scaling arguments and the appearance of logarithms are rather ubiquitous in such a case.

In quantum field theory a result like this will be interpreted as follows: Suppose one performs an experiment to measure some observable quantity (like the binding energy) of the system as well as some of the parameters describing the system (like the coupling constant). If the experiment is performed at a scale corresponding to μ (which, for example, could be the energy of the particles in a scattering cross-section measurement, say), then one will find that the coupling constant that is measured depends on μ. But when one varies μ in an expression like (15), the variation of λ_{phy} will be such that one gets the same value for $\mathcal{E}$.

When you think about it, you will find that it makes a lot of sense. After all the parameters we use to describe our physical system (like λ_{phy}) as well as some of the results we obtain (like the binding energy $\mathcal{E}$ or a scattering cross-section) need to be determined by suitable experiments. In the quantum mechanical problems also one can think of scattering of a particle with momentum k (represented by an incident plane wave, say) by a potential. The resulting scattering cross-section will contain information about the potential, especially the coupling constant λ. If the scattering experiment introduces a (momentum or length) scale μ, then one can indeed imagine the measured coupling constant to be dependent on that scale μ. But we would expect physical predictions of the theory (like $\mathcal{E}$) to be independent of μ. This is precisely what happens in quantum field theory and the toy model above is a simple illustration.

We see from (7) that, in $D = 1$, the coupling constant λ has the dimensions of L^{-1} so there is no difficulty in obtaining $\mathcal{E} \propto \lambda^2$. The one-dimensional integral corresponding to (10) is convergent and you can easily work this out to fix the proportionality constant to be $1/4$. The logarithmic divergence occurs in $D = 2$, which is known as the critical dimension for this problem. The breaking down of naive scaling arguments and the appearance of logarithms are rather ubiquitous in such a case. (There are other fascinating issues in $D \geq 3$ and in scattering but that is another story.)

The examples discussed here are all explored extensively in the literature and a good starting point will be the references [1-5].

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Suggested Reading

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