Exploring Black Hole Physics via Dimensional Analysis

Praveen Pathak[1]
Vinay Uppal[2] and
Vijay A. Singh[1]
[1] Homi Bhabha Centre For Science Education (TIFR),
V. N. Purav Marg, Mankhurd,
Mumbai 400 088, India.
[2] Physics Department
Indian Institute of Technology
Bombay, Mumbai 400 076
India.

Dimensional analysis is a powerful instrument in the physicist's arsenal. It not only helps to check equations, but can also be used to derive important ones. We illustrate this by deriving some important expressions in black hole physics. Starting with a simple input, namely that the properties of an uncharged, non-rotating black hole depend only on the mass (the '*no hair theorem*') we employ dimensional analysis to derive expressions for the area of the event horizon, the entropy, the evaporation time and the Hawking temperature, among others. Our derivations are accessible to pre-college students, but its exotic implications may leave them pondering for a very long time.

1. Introduction

Every beginning student in physics is enjoined to employ dimensional analysis. Dimensional analysis is basically a technique to study the interrelationship between a physical quantity and other variables. The physical quantity to be studied is expressed in terms of a fundamental set of dimensions usually taken to be mass (M), length (L), time (T), temperature (K), etc. However these base dimensions $\{M, L, T, K\}$ are not sacrosanct. For mechanical quantities the base dimensions could be $\{F, L, T\}$ or even $\{\rho, L, T\}$, where F is the force and ρ is the mass density. The existence of equivalent, irreducing base quantities has been known for long and is codified in a famous theorem called the Buckingham Π [1]. Further one could use another set, for example the classical and quantum mechanical constants in physics. These classical constants could be the gravitational constant (G), the speed of light (c), and the Boltzmann constant (k_{B}). On the other hand Bohr

Keywords.
Dimensional analysis, black hole, entropy, event horizon, Bekenstein conjecture, Hawking radiation.

Table 1. Dimensions and numerical values in SI system of some of the classical and quantum fundamental constants.

Constants	Dimensions	Numerical Value
h	ML^2T^{-1}	6.63×10^{-34} J-s
k_B	$ML^2T^{-2}\theta^{-1}$	1.38×10^{-23} J-K^{-1}
c	LT^{-1}	3.00×10^8 m-s^{-1}
G	$M^{-1}L^3T^{-2}$	6.67×10^{-11} m^3-kg^{-1}-s^{-2}

magneton (μB) and Planck's constant (h) are quantum mechanical constants. The base dimensions and numerical values of these constants are listed in *Table* 1.

Let us take an example. For a black body the energy radiated per unit time is given by Stefan–Boltzmann's law as

$$Q = \sigma A T^4.$$

Here T is the temperature of black body, A is its surface area, and σ is the Stefan–Boltzmann constant ($\sigma = 5.67 \times 10^{-8}$ J-s^{-1}-m^{-2}-K^{-4}). Hence σ has the dimensions of energy per unit area per unit time per fourth power of temperature. It can be written in terms of basic dimensions as

$$[\sigma] = MT^{-3}K^{-4} \tag{1}$$

Note that symbol [ ] represents dimensions. However σ can also be expressed in terms of fundamental constants as

$$[\sigma] = [G]^x[c]^y[h]^z[k_B]^w$$

To obtain the values of $x,\ y,\ z$ and w, we put dimensions of $G,\ c,\ h$, and k_B from *Table* I on the right hand side and equate it to the left hand side dimensions of σ from (1). This yields

$$[\sigma] = \frac{{k_B}^4}{c^2h^3}. \tag{2}$$

There are two things we should note about this. First, the appearance of h indicates the quantum nature of Stefan–Boltzmann constant. This is to be expected since

Stefan's law can be derived from the more basic Planck radiation law. Second, the numerical value of right hand side (1.38×10^{-9}) is different from observed value. The inclusion of a dimensionless constant can fix this discrepancy.

One must note that dimensional analysis has proved to be a powerful tool in many situations where a theoretical derivation and explanation is difficult or often impossible. Let us say you may want to determine the time period of a simple pendulum. You will do many experiments changing the two variables: l, length of the pendulum and m, mass of the pendulum. You may also realize that gravity will play a role and hence you will include acceleration due to gravity g ($[g] = LT^{-2}$). Thus you may express time period t_p as

$$[t_p] = m^x l^y g^z.$$

An elementary analysis yields $x = 0,\ y = -z = 1/2$. Thus

$$t_p = a\sqrt{\frac{l}{g}},$$

where a is a dimensionless constant which can be trivially fixed to be 6.28 ($\approx 2\pi$) by examining the data. Note that mass is an irrelevant variable and this fact comes out naturally in the dimensional analysis. Admittedly we have taken a simple example. In what follows we will demonstrate the power of dimensional analysis to explore the physics of black hole[1].

A black hole is a superdense object which may be formed by the collapse of a massive star. In the process its gravitational field becomes so intense that its escape velocity exceeds the speed of light.

[1] Black hole was one of the themes in the International Physics Olympiad 2007. The Olympiad was held in Isfahan, Iran from July 13–22, 2007.This was our tenth participation in this prestigious event. A total of sixty-nine nations participated. Our performance was a success and we secured medals which included *Two Gold*, *Two Silver* and *One Honorable Mention*.

2. Black Hole Physics

A black hole is a superdense object which may be formed by the collapse of a massive star. In the process its gravitational field becomes so intense that its escape velocity exceeds the speed of light. Further, its intense gravity sucks in nearby matter. Because of this no object, not even electromagnetic radiation, can escape from it. Hence the name 'Black Hole'.

Every black hole is surrounded by an 'event horizon'. Roughly speaking event horizon is the boundary of the black hole. Light emitted from within the event horizon can never reach an outside observer.

Let us take the example of our Sun. From energy conservation one can easily show that escape velocity from the surface of Sun is given by

$$v_{esc} = \sqrt{\frac{2GM}{R}}.$$

Here R is the radius and M is the mass of the Sun. If the Sun with its present mass (2×10^{30} kg) becomes a black hole i.e., $v_{esc} = c$, then a simple calculation shows that its radius would be 3 km whereas its current radius is about 7×10^5 km! It is believed that a galaxy has a black hole at its center.

Every black hole is surrounded by an 'event horizon'. Roughly speaking event horizon is the boundary of the black hole. Light emitted from within the event horizon can never reach an outside observer. Further, anything that passes from the external observer's side through the event horizon cannot be retrieved.

There is an important theorem in black hole physics called 'no hair theorem'. The no hair theorem, more accurately no hair conjecture, implies that an uncharged, non-rotating black hole is *completely characterized by it's mass.* Consider two black holes having the same mass. One is composed of matter: say proton, neutron, and electron. The other of antimatter: say antiproton, antineutron, and positron. To an observer outside the event horizon these two would be completely indistinguishable. In the words of the physicist John Wheeler: "Black holes have no hair." We will use this theorem judiciously along with dimensional analysis to unravel the physics of black holes. In what follows we will *adopt a question and answer format.*

1. Let A be the area of an event horizon. Using dimensional analysis, express A in terms of constants G, c and the mass m of the black hole.

Solution: We write

$$[A] = [G]^x [c]^y [m]^z.$$

Area A has dimensions of L^2. From *Table* I we put the dimensions of G and c in the above equation. Equating the order of base dimensions M, L, and T we obtain

$$\begin{aligned} -x + z &= 0, \\ 3x + y &= 2, \\ -2x - y &= 0. \end{aligned}$$

Solving the equation yields $x = z = 2$ and $y = -4$. Hence

$$A = G^2 c^4 m^2. \tag{3}$$

In what follows we assume that the surface area of a black hole is the same as the area of its event horizon (A).

In 1971, using classical general relativity, Hawking proved that the total area of the event horizons of two collapsing black holes is greater than the sum of the individual ones [2]. In other words area A of any black hole can never decrease in any process ($\Delta A \geq 0$). This statement is analogous to the second law of thermodynamics, where we say that the total entropy of a closed system never decreases in any process. Bekenstein used this analogy to define the entropy of a black hole [3]. If a particle goes into a black hole, it disappears. Information is lost during the process. For example, we do not have any idea about the status of the particle once it disappears. But according to Hawking there is always an increase in the black hole's area which results from the disappearance of the particle. The lost information is in fact related to the surface area. Disappearance of the particle into the black hole invariably results in an increase in the latter's area. Hence Bekenstein argued that the entropy of a black hole is proportional to its surface area. Thus

$$S = \eta A. \tag{4}$$

In 1971, using classical general relativity, Hawking proved that the total area of the event horizons of two collapsing black holes is greater than the sum of the individual ones.

Here η is a constant.

2. Using basic dimensions of entropy, express η in terms of fundamental constants $h,\ c,\ G$, and $k_{\rm B}$.

Solution: We know that entropy is

$$\mathrm{d}S = \frac{\mathrm{d}Q}{\theta}, \tag{5}$$

where dQ is the heat exchanged during the process and θ is the absolute temperature of the system. Equation (5) gives the dimensions of entropy to be $[S] = ML^2T^{-2}K^{-1}$. We have

$$\begin{aligned} \eta &= S/A \\ \text{and} \quad [\eta] &= [S][A]^{-1} = MT^{-2}K^{-1}. \end{aligned} \tag{6}$$

To express η in terms of fundamental constants we write

$$[\eta] = [G]^x[h]^y[c]^z[k_{\rm B}]^w = \\ M^{-x+y+w}L^{3x+2y+z+2w}T^{-2x-y-z-2w}K^{-w}. \tag{7}$$

Equating (6) and (7) yields

$$\begin{aligned} -x+y+w &= 1, \\ 3x+2y+z+2w &= 0, \\ -2x-y-z-2w &= -2, \\ w &= 1, \\ \text{which gives} \quad x = y = -w &= -1,\ z = 3. \end{aligned}$$

Thus

$$\eta = \frac{c^3k_{\rm B}}{Gh}. \tag{8}$$

Using (3) and (8)

$$S = \frac{Gk_{\rm B}}{ch}m^2. \tag{9}$$

Note that equation (9) is similar to the entropy–area relation obtained by Bekenstein [3] except for a numerical factor $(\ln 2)/4$. Later Hawking showed that $S = A/4$ if we set fundamental constants to unity. The fact that the expression for S contains an h should not surprise us because it appears in the formulae for entropy of many thermodynamic systems that are conventionally regarded as classical, for example, the Boltzmann ideal gas.

The inclusion of entropy in black hole physics indicates that the black hole cannot be at zero temperature.

It was believed that since a black hole absorbs all the radiation falling on it, it should be at temperature 0 K. Or in other words it cannot emit radiation. But the inclusion of entropy in black hole physics indicates that the black hole cannot be at zero temperature. Using quantum mechanics Hawking showed that "black holes are not quite black" [4] or in other words black holes emit particle and radiation. The radiation is thermal radiation similar to black-body radiation [5]. The temperature corresponding to the radiation is called the Hawking temperature ($\theta_{\rm H}$).

3. Use relation $E = mc^2$ to define the energy of a black hole in terms of mass. Analogous to the laws of thermodynamics use entropy relation (9) to express the Hawking temperature $\theta_{\rm H}$ in terms of mass and other fundamental constants. Assume that the black hole does no work on the surroundings.

Solution: From the first law of thermodynamics

$$\mathrm{d}E = \mathrm{d}Q + \mathrm{d}W.$$

Also $\mathrm{d}S = \mathrm{d}Q/\theta_{\rm H}$. Setting the 'work terms' $\mathrm{d}W$ to zero [4] and equating the above equations

$$\mathrm{d}E = \theta_{\rm H}\,\mathrm{d}S. \tag{10}$$

From $E = mc^2$, $\mathrm{d}E = c^2\mathrm{d}m$.
Equation (9) gives[2]

$$\frac{\mathrm{d}S}{\mathrm{d}m} = \frac{2Gk_{\rm B}}{ch}m.$$

[2] Since $\mathrm{d}Q = 0$, the use of ordinary derivative in place of partial derivative is permissible.

An isolated black hole slowly loses its mass due to Hawking radiation.

Hence (10) takes the form

$$\theta_{\mathrm{H}} = \frac{\mathrm{d}E}{\mathrm{d}S} = \left(\frac{\mathrm{d}S}{\mathrm{d}E}\right)^{-1} = c^2\left(\frac{\mathrm{d}S}{\mathrm{d}m}\right)^{-1} = \frac{c^3 h}{2Gk_{\mathrm{B}}}\frac{1}{m}. \quad (11)$$

The above expression is similar to the one obtained by Hawking [4]. (See another derviation of the black hole temperature in 'Snippets of Physics' in this issue on page 412.)

4. It is clear from the previous discussion that the mass of an isolated black hole will decrease due to Hawking radiation. Find the rate of change of mass using Stefan–Boltzmann's law. Also find the time t^* that a black hole of mass m takes to evaporate completely.

Solution: From Stefan–Boltzmann's law

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\sigma {\theta_{\mathrm{H}}}^4 A, \quad (12)$$

$$c^2\frac{\mathrm{d}m}{\mathrm{d}t} = -\sigma {\theta_{\mathrm{H}}}^4 A. \quad (13)$$

Putting the expressions for σ, A, and θ_{H} from (2), (3), and (11), we get

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{1}{16}\frac{c^4 h}{G^2}\frac{1}{m^2}. \quad (14)$$

Integrating the above equation

$$\int_m^0 m^2\,\mathrm{d}m = -\int_0^{t^*}\frac{c^4 h}{16G^2}\mathrm{d}t,$$

$$t^* = \frac{16G^2}{3c^4 h}m^3. \quad (15)$$

5. We expect the black hole to be exposed to the cosmic background radiation. Assume that this background radiation is a black-body radiation with a temperature

θ_B and it fills the entire universe. Thus a black hole loses energy through Hawking radiation and gains energy from the cosmic background. Find the mass (m^*) in terms of θ_B and fundamental constants at which this energy exchange reaches equilibrium.

A black hole loses energy through Hawking radiation and gains energy from the cosmic background.

Solution: Energy per unit time received by a black hole is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = c^2\frac{\mathrm{d}m}{\mathrm{d}t} = \sigma({\theta_B}^4 - {\theta_H}^4)A. \qquad (16)$$

Employing expressions from (2), (3) and (11),

$$\begin{aligned} c^2\frac{\mathrm{d}m}{\mathrm{d}t} &= \frac{{k_B}^4}{c^2h^3}\left[{\theta_B}^4 - \left(\frac{c^3h}{2Gk_B}\frac{1}{m}\right)^4\right]\frac{G^2}{c^4}m^2, \\ \frac{\mathrm{d}m}{\mathrm{d}t} &= \frac{G^2m^2}{c^8h^3}(k_B\theta_B)^4 - \frac{hc^4}{16G^2m^2}. \end{aligned}$$

At equilibrium,

$$\left.\frac{\mathrm{d}m}{\mathrm{d}t}\right|_{m=m^*} = 0,$$

which yields

$$m^* = \frac{c^3h}{2Gk_B}\frac{1}{\theta_B}.$$

6. Argue mathematically if this equilibrium is stable?

Solution: At thermal equilibrium

$$\theta_H = \theta_B = c^3h2Gk_B\frac{1}{m^*} = \theta^*$$

From (16) and (11)

$$\text{If} \quad \theta^* > \theta_H \Rightarrow m^* < m, \quad \text{then} \quad \frac{\mathrm{d}m}{\mathrm{d}t} > 0;$$

$$\text{If} \quad \theta^* < \theta_H \Rightarrow m^* > m, \quad \text{then} \quad \frac{\mathrm{d}m}{\mathrm{d}t} < 0.$$

Hence the equilibrium is unstable.

3. Exotic Physics

There are several aspects about the preceding exercises that are worth pondering over. The fact that the Planck constant is associated with the dimensions of the Stefan–Boltzmann constant σ (2) and the entropy (8) has been commented upon. *Table* 2 lists some of the values of black hole properties based on the dimensional analysis carried out in the present article. We have taken the value of the dimensionless constants to be unity and the mass of the black hole to be the solar mass. We find that when the mass of the black hole changes from $M_\odot$ to 1.1 $M_\odot$ ($M_\odot$ stands for solar mass), the corresponding change in entropy is $\approx 10^{51}$ J-K^{-1}. In contrast, for a similar process, the change in entropy for the Sun is $\approx 10^{33}$ J-K^{-1}. This entropy is calculated assuming that the temperature of the Sun is 6273 K, its radius is 6.95 $\times 10^5$ km and it consists of 92% of hydrogen. The phenomenally high value of the entropy is indicative of the highly irreversible character of black hole formation [3]. The Hawking temperature is low (10^{-6} K), but not zero.

Note that we have used Bekenstein conjecture which relates entropy to area (4). It may interest the reader to note that the relationship between perimeter and entropy was employed by Rudolf Peierls to demonstrate the existence of phase transition in the two-dimensional Ising model. It goes by the name of the 'Peierls' argument [6]. Note that the perimeter is the analogue of area in two dimensions.

Table 2. Values of some of the characteristic properties of black holes calculated using solar mass $M_\odot$= 1.98 $\times 10^{30}$ kg. The change in entropy is calculated for a change in mass $M_\odot$ to 1.1 $M_\odot$.

Area of event horizon (A)	G^2m^2/c^4	Eq. (3)	2.17 $\times 10^6$ m^2
η	$c^3k_{\rm B}/Gh$	Eq. (8)	8.43$\times$ 10^{45} JK^{-1}m^{-2}
Change in Entropy (ΔS)	$Gk_{\rm B}m^2/ch$	Eq. (9)	3.84 $\times$ 10^{51} J-K^{-1}
Hawking temperature ($\theta_{\rm H}$)	$c^3h/2Gk_{\rm B}m$	Eq. (11)	4.83$\times$ 10^{-6} K
Time to evaporate (t^*)	$16G^2m^3/3c^4h$	Eq. (15)	1.10 $\times$ 10^{63} years

The mass dependence of some of the physical quantities appears anomalous. For definiteness think of a cube of constant density. Its volume is L^3, where L is its length. Thus its mass scales as L^3. Its surface area is $6L^2$. Hence its area should scale as $m^{2/3}$. However the area of the black hole event horizon scales as m^2. We have seen a sugar cube dissolving in water. This process depends on its area. We can describe this process by an equation analogous to (14), namely

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -a_1 m^{2/3}, \tag{17}$$

where a_1 is a constant. Integrating this equation would yield the dissolution time $t^* \sim m^{2/3}$ in marked contrast to the evaporation time scaling law $t^* \sim m^3$ (15).

Let us examine the expression for the black hole entropy (9). We find that $S \sim m^2$. However the associated Hawking temperature $\theta_{\mathrm{H}} \sim 1/m$ (11). If the black hole mass is large, its temperature is small, and its entropy is large. This is undeniably non-intuitive. Recall that the third law of thermodynamics leads us to believe that as the temperature approaches zero, so does the entropy. One could go further. The specific heat is negative.

$$\begin{aligned} C_V &= \frac{\mathrm{d}E}{\mathrm{d}\Theta}, \\ &= -\frac{2Gk_{\mathrm{B}}}{ch}\, m^2, \end{aligned}$$

where we have used $E = mc^2$ and equation (11). Once again as $m \to 0$, $\Theta_{\mathrm{H}} \to \infty$ and $C_V \to 0$, contrary to our expectations based on classical thermodynamics. Astronomical systems display negative specific heat and there have been attempts to resolve this paradox [7]. A discussion of this paradox would form another article by itself.

Black hole physics is exotic. If by taking recourse to simple dimensional analysis we have been able to create a sense of wonder and mystery about it, then our labour has been well worth it.

Acknowledgment

This work was supported by the National Initiative on Undergraduate Science programme undertaken by the Homi Bhabha Centre for Science Education – Tata Institute for Fundamental Research (HBCSE-TIFR), Mumbai, India. We are also thankful to Dr. Sandip Trivedi (TIFR, Mumbai) and Dr. Bikram Phookun (St. Stephen's College, New Delhi) for useful discussions and suggestions.

Suggested Reading

[1] E Buckingham, On physically similar systems; illustrations of the use of dimensional equations, *Phys. Rev.*, Vol.4, pp.345–376, 1914.

[2] Stephen Hawking, Gravitational radiation from colliding black holes, *Phys. Rev. Lett.*, Vol.26, pp.1344–1346, 1971.

[3] Jacob Bekenstein, Black holes and entropy, *Phys. Rev.* D, Vol.7, pp.2333–2346, 1973.

[4] Stephen Hawking, Particle creation by black holes, C*ommun. Math. Phys*., Vol.43, pp.199–220, 1975.

[5] G S Ranganath, Black-body Radiation, *Resonance*, Vol.13, No.2, 2008.

[6] R E Peierls, On Ising Model of Ferromagnetism, *Proceedings of the Cambridge Philosophical Society*, Vol.32, p.477, 1936.

[7] W Thirring, Systems with negative specific heat, *Z. Phys*., Vol.235, pp.339–352, 1970.