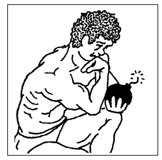
Think It Over



This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

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Problem[1]: Suppose n birds land randomly on a wire connecting two columns, where $n \geq 2$. Each bird is watching its nearest neighbour. What is the expected number of unwatched birds?

[1] Soubhik Chakraborty, *Resonance*, Vol. 13, No.1, p.88, January 2008.

Solution to How Many Birds are Unwatched

Let the length of the wire be w. Let $P_1 \leq P_2 \leq P_3 \leq \cdots \leq P_n$ be the points where the birds landed. From what is given, we may take $(P_1, P_2, P_3, \ldots, P_n)$ to represent n order statistics for a sample of size n drawn from $U(0, w)$ distribution. The joint density of the random variables $P_1, P_2, P_3, \ldots, P_n$ turns out to be

$$f(p_1, p_2, \ldots, p_n)$$

$$= \begin{cases} \frac{n!}{w^n} & \text{for } 0 < p_1 < p_2 < \cdots < p_n < w \\ 0 & \text{otherwise} \end{cases} .$$

The reader is encouraged to supply the proof (see [2]). Now define distances $D_1 = P_1, D_i = P_i - P_{i-1}, i = 2, \ldots, n$, as shown in *Figure* 1.

The inverse transformation is $P_i = D_1 + D_2 + \cdots + D_i$

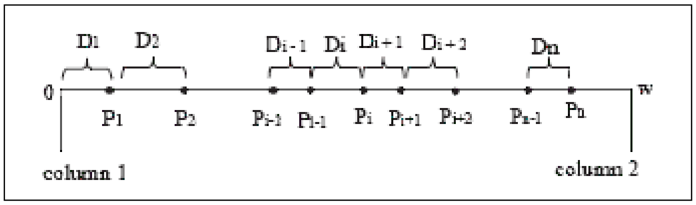

Figure 1.

with Jacobian of transformation as unity (prove!).

Accordingly the joint density of $D_1, D_2, \ldots D_n$ is

$$g(d_1, d_2, \ldots, d_n) =$$

$$\begin{cases} \frac{n!}{w^n} & \text{for } 0 < d_i < w, i = 1, 2, \ldots, n, \sum_{i=1}^{n} d_i < w \\ 0 & \text{otherwise} \end{cases} .$$

Observe that density g is a symmetric function whence D_i's are exchangeable random variables (random variables are called exchangeable if their joint distribution is the same no matter in which order they are observed). For more on exchangeability, see [1].

We now consider two cases:

Case A: $n \geq 4$.

Let us group the birds in three categories.

Category 1: The birds at both ends. There are two such birds.

Category 2: The birds in penultimate positions. There are again two such birds.

Category 3: The remaining $n - 4$ birds.

If $D_2 > D_3$, left most bird is unwatched. Due to exchangeability of the D_i's, we have $P(D_3 < D_2) = P(D_2 < D_3) = \frac{1}{2}$. The penultimate bird is always watched (the end bird watches it!). If $3 \leq i \leq n - 2$, we observe that the ith bird is unwatched when $D_i > D_{i-1}$ and $D_{i+2} < D_{i+1}$. Now $P(D_i > D_{i-1} \cap D_{i+2} < D_{i+1}) = P(D_i > D_{i-1}) P(D_{i+2} > D_{i+1})$ by compound theorem of probability.

Again, due to exchangeability, $P(D_i > D_{i-1}) = P(D_i < D_{i-1}) = \frac{1}{2}$ and $P(D_{i+2} < D_{i+1}) = P(D_{i+2} > D_{i+1}) = \frac{1}{2}$ so that P (ith bird is unwatched, $3 \leq i \leq n-2$) $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. The situation at the right end is similar to that at the left. Since a randomly chosen bird occupies position i with probability $\frac{1}{n}$, it follows from above, using law of total probability, that a randomly chosen bird is unwatched with probability

$$\left(\frac{1}{n}\times\frac{1}{2}\right)+\left(\frac{1}{n}\times 0\right)+\left\{(n-4)\times\frac{1}{n}\times\frac{1}{4}\right\}+\left(\frac{1}{n}\times 0\right)+\left(\frac{1}{n}\times\frac{1}{2}\right)$$

$$= \frac{1}{4}.$$

(Interestingly, this probability is not a function of $n, n \geq 4$)

Next define random variables Y_i's as

$$Y_i = \begin{cases} 1, & \text{if } i\text{th bird is unwatched} \\ 0, & \text{otherwise.} \end{cases}$$

Define probabilities $p_i' = P(y_i = 1), i = 1, 2, \ldots, n$.

We have shown that

$$p_1' = p_n' = \frac{1}{2},$$
$$p_2' = p_{n-1}' = 0,$$
$$\text{and } p_i' = \frac{1}{4}, \quad 3 \leq i \leq n-2.$$

Since the total number of unwatched birds is $Y = \sum_{i=1}^{n} Y_i$, the expected number of unwatched birds is

$$E(Y) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = 2\times\frac{1}{2}+2\times 0+(n-4)\times\frac{1}{4}$$

$$= \frac{n}{4}.$$

Case B: Consider $n \leq 3$.

If $n = 3$, we have two birds of category 1 and one bird of category 2, which is the middle bird.

The probability that a randomly chosen bird is unwatched is

$$\left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times 0\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) = \frac{1}{3},$$

arguing as earlier.

The expected number of unwatched birds is 1 because the middle bird is watched by both the end birds while the middle bird itself will be watching either the bird on left (rightmost bird is unwatched) or the bird on right (leftmost bird is unmatched) whichever is nearer. The case $n = 2$ is trivial. There are two birds of category 1, each watching the other leading to expectation of unwatched birds being zero.

Suggested Reading

[1] **Sheldon Ross, *A First Course in Probability*, 6th ed., Pearson Edu,**

[2] **R Hogg and A Craig, *Introduction to Mathematical Statistics*, 3rd ed., The Macmilland Co. of Canada, Toronto, 1970.**

[3] ***Amer. Math. Monthly*, Vol.89, pp.274–275, 1982.**