In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Spinning Ball Flight Under Moderate Wind

Predicting the flight of spinning balls is an exciting aspect while playing cricket, tennis, table tennis or soccer. These games demand a wide range of skills to exploit the aerodynamic effects induced by spin, wind and gravity. The theory underlying these aerodynamic effects unveils bizarre opportunities for budding cricket, tennis and soccer stars!

Keywords
Magnus effect; cycloid, spiroid, terminal velocity, sine/cosine integral, leg spin, off spin, s-spin, top spin, side spin, backward spin.

1. Introduction

A previous classroom article [1] discussed the effect of strong wind on tennis ball flight. However, more commonly, moderate winds prevail during playtime. Also bad shots hit by batsmen off the edge of the bat induce spin to the ball. The combined effect of spin, drag and gravity produce spectacular effects in tennis ball cricket and soccer. As discussed in [1], the governing differential equations of motion under moderate wind condition are a formidable set of coupled non-linear differential equations! In general, the nonlinear equations defy analytical strategies and demand numerical methods for their solutions. In this article, we revisit these equations, and re-examine the special case of the ball returning to the point of projection. For the sake of continuity, we recaptulate the main ideas from the previous classroom article on tennis ball flight without spin under strong wind.

2. Flight Without Spin: Strong Winds

Suppose a batsman hits a tennis ball. Let the ball velocity be $\mathbf{v}^{bg}$, where superscripts denote the velocity of the ball with respect to the ground. Assuming a steady wind velocity $\mathbf{v}^{wg}$, the relative velocity of the ball with respect to the wind is $\mathbf{v}^{bw} = \mathbf{v}^{bg} - \mathbf{v}^{wg}$. In general, $\mathbf{v}^{bg}$ and $\mathbf{v}^{bw}$ have all three components while $\mathbf{v}^{wg}$ has only two components parallel to the cricket ground. The drag force varies as the square of the relative velocity magnitude $(v^{bw})^2$ and acts in the direction opposite to the vector $\mathbf{v}^{bw}$. In vector notation, the drag expressed per unit mass is

$$\frac{\mathbf{D}}{m} = -k(v^{bw})^2\frac{\mathbf{v}^{bw}}{v^{bw}} = -kv^{bw}\mathbf{v}^{bw}. \qquad (1)$$

In equation (1) k is the drag coefficient, which depends on the shape, size, orientation, speed and texture (fur, moisture, etc.,) of the flying object. It is convenient to introduce a standard wind speed c_o , which produces a drag equal to the ball weight. For a standard 60 g tennis ball, c_o is about 25 m/s. Thus,

$$\mathbf{D} = -mg\frac{v^{bw}}{c_o^2}\mathbf{v}^{bw}.$$

We are now well armed to attack the vector equation of motion:

$$-m\mathbf{a}^{bg} = (\mathbf{D} + m\mathbf{g}), \qquad (2)$$

where $\mathbf{a}^{bg}$ is the acceleration vector. In order to fix the directions of the unit vectors, we take the batsman as the origin of co-ordinates (X,Y,Z). The X-axis is along the pitch, Y-axis is pointing in the direction of the leg umpire, and Z-axis pointing to the sky above the batsman. The wind vector is $\mathbf{v}^{wg} = (\mathbf{i}v_x^{wg} + \mathbf{j}v_y^{wg})$ and the acceleration due to gravity is –$\mathbf{k}$g. The vector equation of motion looks deceptively short and simple, but wait until you read about all the complications caused by the $\mathbf{D}$ term! Expanding this equation into its scalar components.

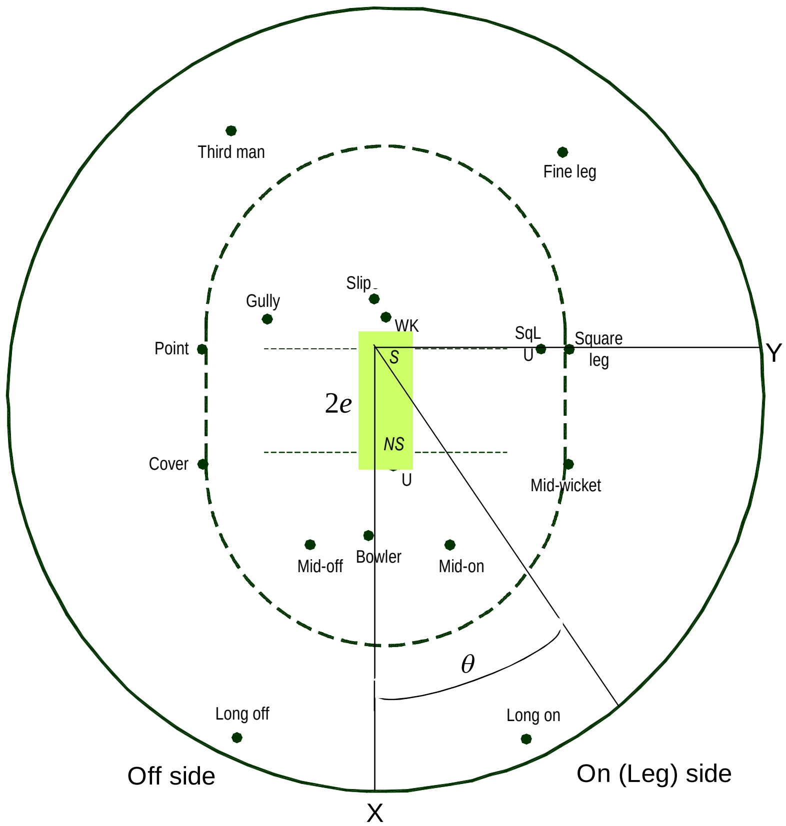
$$\begin{aligned} a_x^{bg} &= -g\frac{v_x^{bw}v^{bw}}{c_0^2}, \\ a_y^{bg} &= -g\frac{v_y^{bw}v^{bw}}{c_0^2}, \\ a_z^{bg} &= -g\left[1+\frac{v_z^{bw}v^{bw}}{c_0^2}\right]. \end{aligned} \tag{3}$$

Clearly, there is no hope for an easy solution considering that hardly anything is known about the variation of $\mathbf{a}^{bg}$ and $\mathbf{D}$ with time in the three-dimensional, wind-blown space. It seems like it is not good cricket! But, wait! For strong winds $v^{bw} \approx -v^{wg}$! Tennis ball cricket does not stop because of a gale or two! Now, under this gale force, the formidable nonlinear coupled equations bow down to simple decoupled linear equations:

$$\begin{aligned} a_x^{bg} &= -g\frac{v_x^{bw}v^{bw}}{c_0^2}, \\ a_y^{bg} &= -g\frac{v_y^{bw}v^{bw}}{c_0^2}, \\ a_z^{bg} &= -g. \end{aligned} \tag{4}$$

It is indeed remarkable that strong winds make the going smooth by way of decoupling the maze of coupled non-linear differential equations into a docile set of three linear equations! To keep things even more simple, we assume that the wind is blowing along the direction of the pitch into the batsman along the negative X-axis. Further, we assume the ball is hit either high over the

Box 1. Cricket: Fielding Positions for a Right-handed Batsman.



This map shows the batsman (S), straight umpire (U), square leg umpire (Sq L U), and fielder positions. The tussle between the bowler and the batter is orchestrated by the captain and his team of players positioned strategically in the oval field. The ellipse with major axis 2a has the batter and the bowler at the foci separated by 2e. The polar equation of the ellipse using the batter as the origin is $r = p/(1-\varepsilon\cos\theta)$, where p is the boundary distance along Y-axis and ε is the ratio e/a. Tennis ball cricket, however, bends geometric rules to blend in with the available space. The sprawling city of Bangalore and suburbs can perhaps boast of the largest tennis ball cricket following in the world with well over a million players. This revolution was fuelled in the 1960s with Test cricket legends like B S Chandrashekar and G R Vishwanath participating actively in suburban tournaments. There is no doubt that tennis ball cricket represents the most imaginative Indian innovation in sports and pastime in recent times. (Adapted from http://en.wikipedia.org/wiki/fielder)

bowler or pulled low over the square (see *Box* 1) leg umpire SqLU. In the first case the ball balloons up over the bowler and the wind drags it back towards the batsman. Curious? Let us derive a formula to drive home this idea. The docile set of three equations becomes a sweet set of two:

$$a_x^{bg} = -g\left(\frac{v_x^{wg}}{c_0}\right)^2,$$
$$a_z^{bg} = -g.$$

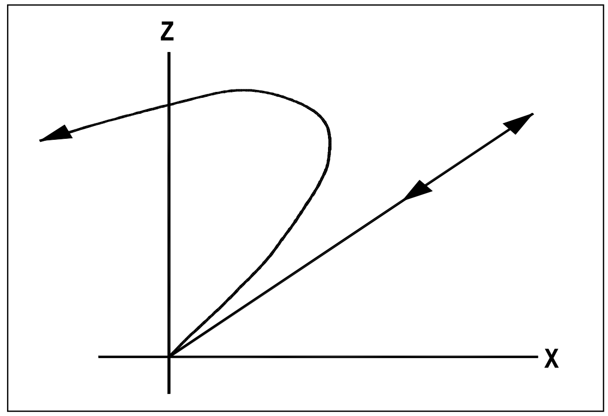


Figure 1. Flight paths in vertical plane under strong wind blowing left.

Omitting superscripts and integrating the equations

$$v_x = v_{xo} - \alpha g t,$$
$$v_z = v_{zo} - g t,$$

where $\alpha = [\frac{v_x^{bw}}{c_0}]^2$; and, v_{zo}, v_{xo} are the initial velocity components. Integrating again and eliminating the time variable t gives the flight path. This was proved to be given by the equation

$$\frac{(x - \alpha z)^2}{z v_{x0} - x v_{z0}} = \frac{2(\alpha v_{z0} - v_{x0})}{g}. \quad (5)$$

This equation represents a tilted parabola. The angle of tilt is given by $\arctan(1/\alpha)$ with respect to the ground. The wind effect is like playing on a mountain slope. Thus, strong winds and level playing grounds do not go together! Some typical trajectories are shown in *Figure* 1. Observe the strange but special case of the ball moving up and down a straight line. In this extraordinary situation the wind returns the ball back to the bat. This is not as odd as it seems when there is no wind. A ball thrown vertically up comes down to the same point eventually. Under light breeze conditions, this special angle is about 80 degrees, and under strong winds angles as low as 45 to 60 degrees are possible. Under moderate breeze, however the ball executes a breezy loop resembling an airfoil shape.

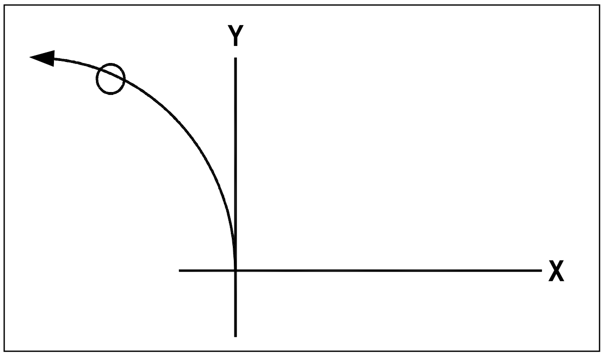


Figure 2. Aerial view of swinging ball under strong wind blowing left.

We concluded the previous classroom article [1] with a second example of a tennis ball hit low over the square leg umpire. We are now interested in the way the ball drifts in the direction of the wind when viewed from the top in the XY plane. Therefore, we need only two equations.

$$\begin{aligned} a_x^{bg} &= -g\left(\frac{v_x^{wg}}{c_0}\right)^2, \\ a_y^{bg} &= 0. \end{aligned}$$

Integrating after dropping superscripts, we get

$$\begin{aligned} v_x &= \alpha g t, \\ v_y &= v_{y0} = \text{const.} \end{aligned}$$

In this case, the wind carries the ball behind the position of the leg umpire before the ball hits the ground. Looking from above, the ball sweeps a parabolic arc in the XY-plane (*Figure* 2), given as follows

$$y^2 + \left(\frac{2v_{yo}^2}{\alpha g}\right) x = 0, \tag{6}$$

3. Spinning Ball Flight: Moderate Winds

Inspired by the legendary free kick executed by Beckham on 26th June 1998, Cook and Goff [2] explored the combination of parameters for successful soccer kicks. Their

The magnus force is directly proportional to the cross product of spin(**s**) and relative velocity of the ball with respect to air (**R**).

calculations suggest that Beckham kicked the soccer ball at about 100 km/h with a spin of 600 rpm! This kick rose well over a bewildered wall of defenders enroute the top edge of the goal (60 ft yonder, 8 ft up and 12 ft to the left). There is a trade off between spin and translation while bending the soccer ball in flight. In tennis ball cricket, however, spinning shots occur by chance rather than by design. The additional spin induced aerodynamic force is the famous magnus effect. The magnus force is directly proportional to the cross product of spin($\mathbf{s}$) and relative velocity of the ball with respect to air ($\mathbf{R}$). The constant of proportionality k_s, like the drag coefficient, depends on the shape, size and texture of the ball. This additional aerodynamic force gives the new equation of motion,

$$m\mathbf{a}^{bg} = (\mathbf{D} + m\mathbf{g}) + k_s\mathbf{s} \times \mathbf{R}. \tag{7}$$

Drag force is proportional to the square of relative velocity and acts opposite in direction to the relative velocity vector. We take c_w as the wind speed and when the wind speed attains c_o, the drag becomes equal to the weight of the ball. Thus, c_o is also the terminal velocity of the ball falling freely in the vertical direction. With respect to the spinning ball, a spin of magnitude s_o and wind speed c_o produce a magnus force of mg. Based on Cook and Goff data for soccer ball[2], c_o is about 25 m/s and s_o about 600 rpm. It is rather a remarkable coincidence that c_o for a soccer ball is about the same as for a tennis ball!

The batsman is at the origin of coordinates (X,Y,Z); X-axis is along the pitch and wind is blowing in the negative X direction (i.e., towards the batsman). We take Y-axis in the vertical direction in the sequel; and, therefore, the Z-axis is now pointing away from the leg umpire. However, we do not really need any specific value for c_0 if we use non-dimensional parameters. The relative velocity of the ball with respect to the wind is denoted as $R = [(u + c_w)^2 + v^2 + w^2]^{1/2}$, where u, v, w represent the ball velocity components. The governing equations of motion of a spinning ball of mass m are:

$$\begin{aligned} m\frac{\mathrm{d}u}{\mathrm{d}t} &= -k_d(u + c_w)R + k_s(s_2w - s_3v), \\ m\frac{\mathrm{d}v}{\mathrm{d}t} &= -k_dvR - mg + k_s(s_3(u + c_w) - s_1w), \\ m\frac{\mathrm{d}w}{\mathrm{d}t} &= -k_dwR + k_s(s_1v - s_2(u + c_w)). \end{aligned} \quad (8)$$

In the above equations s_1, s_2, s_3 are the spin components about X, Y and Z axes, respectively. It is convenient to normalize the drag and magnus force with respect to the weight of the ball by normalizing the velocity components with respect to c_o and spin components with respect to s_o. The non-dimensional form of equations of a spinning ball becomes

$$\begin{aligned} \frac{1}{g}\frac{\mathrm{d}u}{\mathrm{d}t} &= -(u + c_w)R/{c_o}^2 + (s_2w - s_3v)/c_os_o, \\ \frac{1}{g}\frac{\mathrm{d}v}{\mathrm{d}t} &= -(vR + {c_o}^2)/{c_o}^2 + (s_3(u + c_w) - s_1w)/c_os_o, \\ \frac{1}{g}\frac{\mathrm{d}w}{\mathrm{d}t} &= -(wR)/{c_o}^2 + (s_1v - s_2(u + c_w))/c_os_o. \end{aligned} \quad (9)$$

The above set resembles Volterra–Lotka equations used for atmospheric and ecological modelling. These equations are often expressed in the Kolmogorov form:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = uf(u, v, w), \ \ \frac{\mathrm{d}v}{\mathrm{d}t} = vg(u, v, w), \ \ \frac{\mathrm{d}w}{\mathrm{d}t} = wh(u, v, w).$$

Lorenz developed similar equations to open the doors of nonlinear dynamics and chaos [4]. The general case of the spinning ball flight requires 4 initial velocities (u_o, v_o, w_o, c_w) and three spin components (s_1, s_2, s_3). We assume in this formulation that there is no decay in spin with time ($|\mathbf{s}|$ = constant).

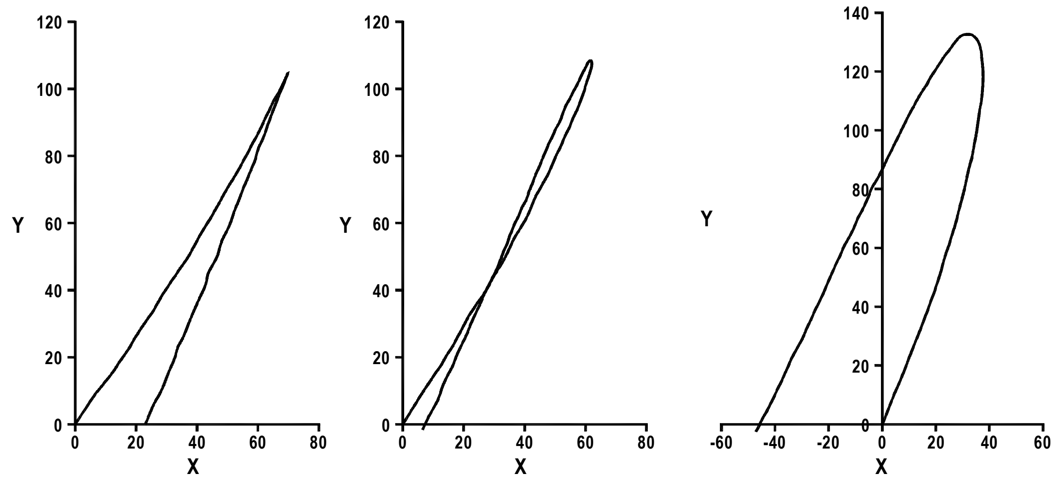


Figure 3. Flight paths for increasing projection angle (gentle wind blowing left).

4. Spinless Flight: Moderate Winds

In this section, we reconsider the flight without spin in the vertical (XY) plane. Once again, the idea is to explore the possibility of the ball returning to the origin for a given set of values (u_o, v_o, c_w). Unlike the tennis ball flight under strong wind, flight under moderate wind offers limited theoretical scope for deriving closed form solutions. It becomes neccessary to explore numerical strategies to gain insight into the flight of the tennis ball. For the sake of providing a physical feel for the problem, we illustrate the rich variety of flight paths by assuming $c_w = 30$ m/s , $c_o = 44.3$ m/s, $R_o = 100$ m/s in *Figure* 3.

There is a spectacular diversity of flight paths possible depending on the initial angle of projection (*Figure* 3). Observe the case of a looping flight path. This loop gradually evolves into a cusp. At a special angle of projection, the tennis ball returns to the origin. Above this special angle the tennis ball falls behind the batsman (and hopefully not into the wicketkeepers hands!).

To express results using non-dimensional quantities, we can take c_o as the unit of speed. We define non-dimensional velocity $r = (u^2 + v^2)^{1/2}$. The unit of length is

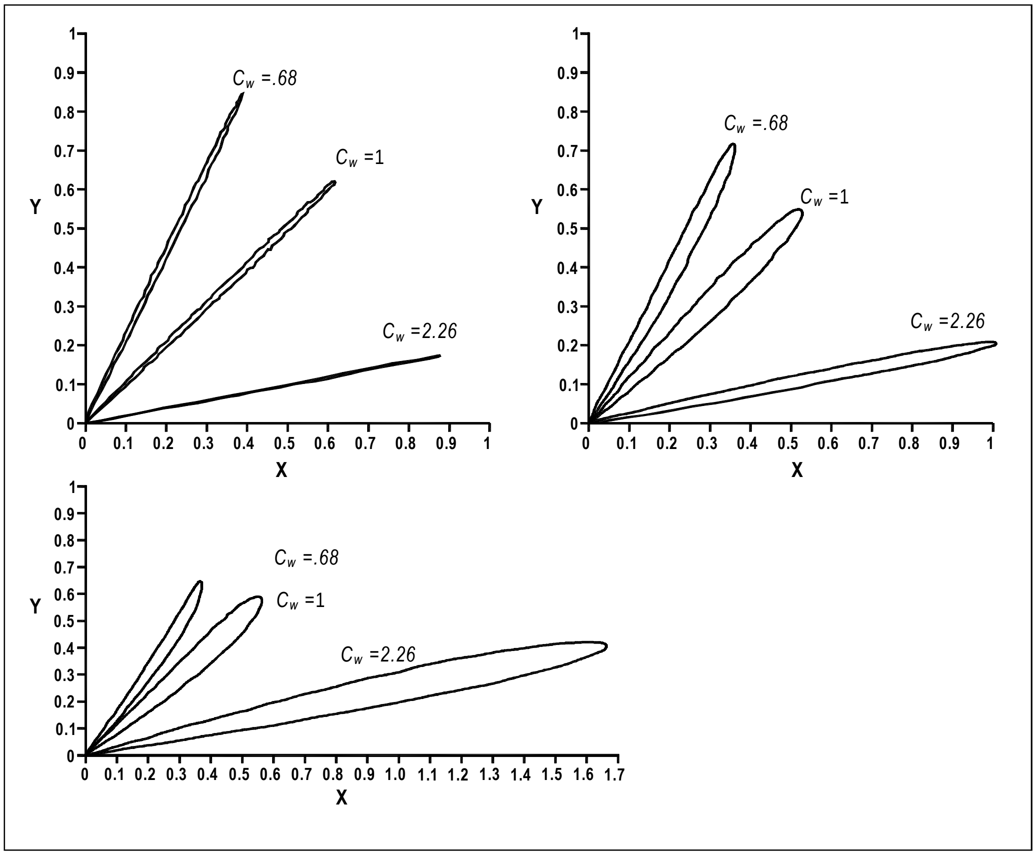


Figure 4. Return Paths [r_o = 0.226, θ_o = 64.2°, 43.5°, 10.75°], [r_o = 2.26, θ_o = 56.3°, 38.7°, 9.69°], [r_o = 22.6, θ_o = 52.03°, 37.2°, 9.64°].

taken as h_o, the height attained by the ball when thrown vertically with a velocity R_o assuming an effective value of gravity $g^* = g(\sin\theta_o + \alpha\cos\theta_o)$, where θ_o is the angle of projection and $\alpha = {c_w}^2/{c_o}^2$. When there is no wind ($c_w = 0$), it can be shown that $h_o = \frac{{c_o}^2}{2g^*}\ln(1+\frac{{R_o}^2}{{c_o}^2})$, for vertical flight.

Figure 4 presents non-dimensional trajectories for three different non-dimensional wind speeds $c_w = 0.68, 1.0, 2.26$. It is interesting to observe the loops getting narrower for decreasing values of r_o. For vanishingly small r_o, the special angle $\theta_s = \pi/2 - \alpha$ as discussed in [1] for strong winds. The return path aligns with the terminal velocity vector. The terminal velocity vector is vertical when there is no wind. The concept of terminal velocity in steady head wind implies that the ball is carried by the wind ($u = -c_w$). The vertical component of the terminal velocity is c_o. Therefore, the terminal angle θ_t satisfies $\tan\theta_t = (c_o/c_w)$. The angles θ_s, θ_t and the initial angle of projection θ_o complete the kinematic picture of the flight.

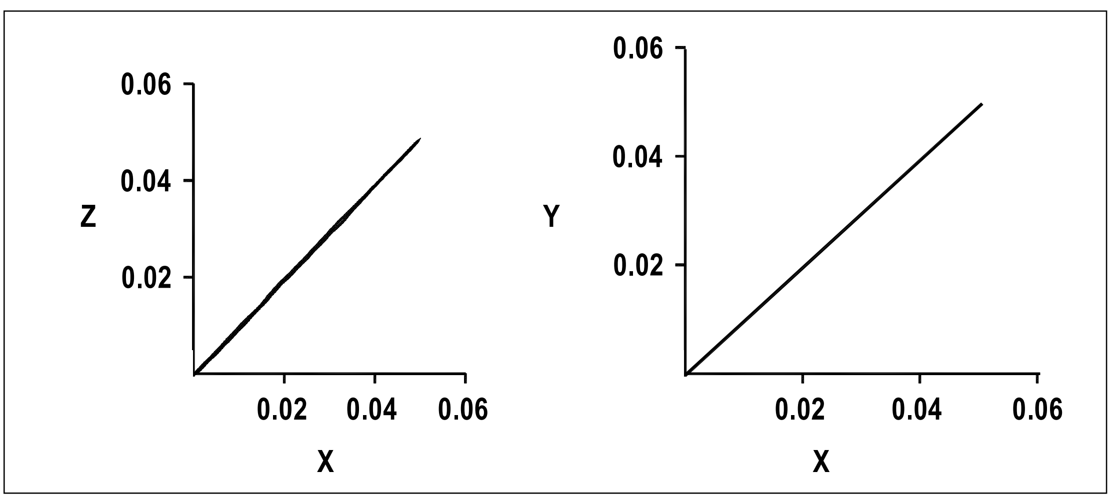
5. Spinning Ball: Strong Winds

It is interesting to study the flight of spinning ball under strong wind because the equations simplify enormously. Under strong wind $u, v, w << c_w$; and $s_1 = s_3 = 0$,

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= -g\frac{{c_w}^2}{{c_o}^2},\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -g,\\ \frac{\mathrm{d}w}{\mathrm{d}t} &= -g\frac{s_2 c_w}{c_o s_o}.\end{aligned} \tag{10}$$

This case is similar to the tennis ball flight under strong wind except that the path of the ball is not in the XY-plane (see *Figure* 5 for the special case of ball returning to the origin). The motion of the ball would be executed on a straight line inclined equally to the coordinate axes, if $u_o = v_o = -w_o = .0226c_o$; $c_w = c_o = s_2$. The result is shown in the top and side views in *Figure* 5.

Figure 5. Spinning ball returning to origin; $u_o = v_o = -w_o = .0226c_o$; $c_w = c_o = s_2$.



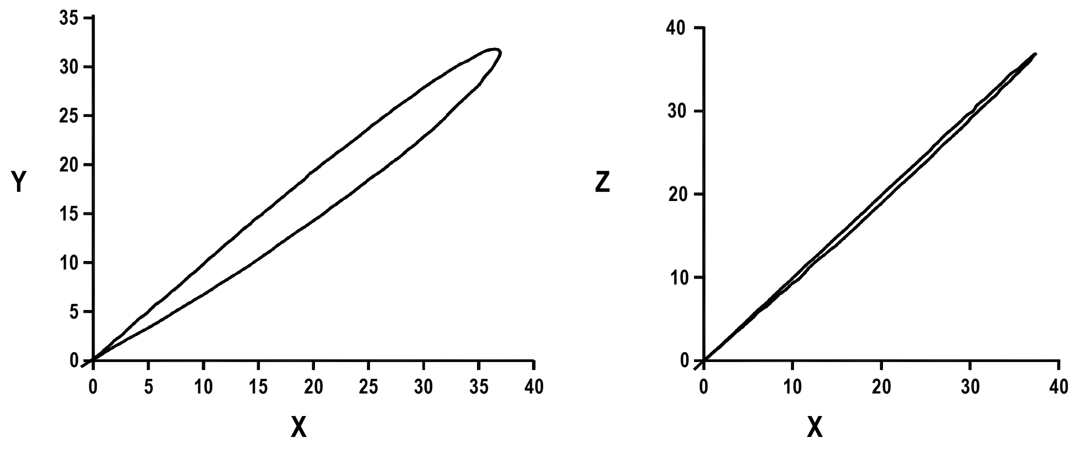


Figure 6. Spinning ball returning to the origin: Moderate winds.

6. Spinning Ball: Moderate Winds

When the ball and wind speeds are of the same order, the trajectory of the ball is shown in *Figure* 6. The ball executes a 3-D loop before returning to the origin for the conditions $u_o = v_o = -w_o = c_o$, $c_w = c_o$. This flight trajectory has constantly changing curvature and torsion.

7. Spinning Ball: Aerial View

Wind and gravity complicate matters. There are situations where gravity and wind become relatively unimportant. The absence of gravity and wind, as we will see, brings out a rich diversity of flight paths as with frisbees and boomerangs. Let us imagine that the ball is in a gravity- and wind-free environment. Let us throw the ball spinning about the Y-axis in the XZ-plane and along the X-axis with $v = w = 0$. Let $s_2 \neq 0, s_3 = s_1 = 0$. The equations governing the motion of the ball reduce to:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= -\frac{uR}{{c_o}^2}g + g\frac{s_2 w}{c_o s_o}, \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= -\frac{wR}{{c_o}^2}g - g\frac{s_2 u}{c_o s_o}. \end{aligned} \qquad (11)$$

The absence of gravity and wind as we will see, brings out a rich diversity of flight paths as with frisbees and boomerangs.

Putting $u = R\cos\theta$ and $w = R\sin\theta$, we can show that

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -g\frac{R^2}{{c_o}^2}.$$

Upon integration we get

$$R(t) = R_o\frac{{c_o}^2}{{c_o}^2 + R_o g t}.$$

The corresponding solution for $\theta(t) = -\frac{g s_2}{c_o s_o}t$ is as follows

$$\begin{aligned} u &= R_o\frac{{c_o}^2}{({c_o}^2 + R_o g t)}\cos(-\frac{g s_2}{c_o s_o}t), \\ w &= R_o\frac{{c_o}^2}{({c_o}^2 + R_o g t)}\sin(-\frac{g s_2}{c_o s_o}t). \end{aligned} \quad (12)$$

Taking $a = (c_o s_2)/(R_o s_o)$ and $c = (g s_2)/(c_o s_o)$, the flight path in the XZ-plane is given in terms of sine and cosine integrals.

$$\begin{aligned} x(t) &= \frac{{c_o}^2}{g}((Si(a+ct) - Si(a))\sin(a) \\ &\quad +(Ci(a+ct) - Ci(a))\cos(a)), \\ z(t) &= \frac{{c_o}^2}{g}((Ci(a+ct) - Ci(a))\sin(a) \\ &\quad -(Si(a+ct) - Si(a))\cos(a)). \end{aligned} \quad (13)$$

In the above solution $Si(x)$ and $Ci(x)$ are the definite integrals between the limits 0 to x of the functions $\frac{\sin(x)}{x}$ and $\frac{\cos(x)}{x}$, respectively. We notice the gradual decay in speed with time as the ball spirals towards its final position. This final position depends on the initial velocity and spin. The spiral variation of the velocity is portrayed in *Figure* 7. Note that the velocity of the spinning ball varies inversely with time elapsed vanishing eventually. The spinning ball approaches its final destination with coordinates (68.7,–124.3)m when the

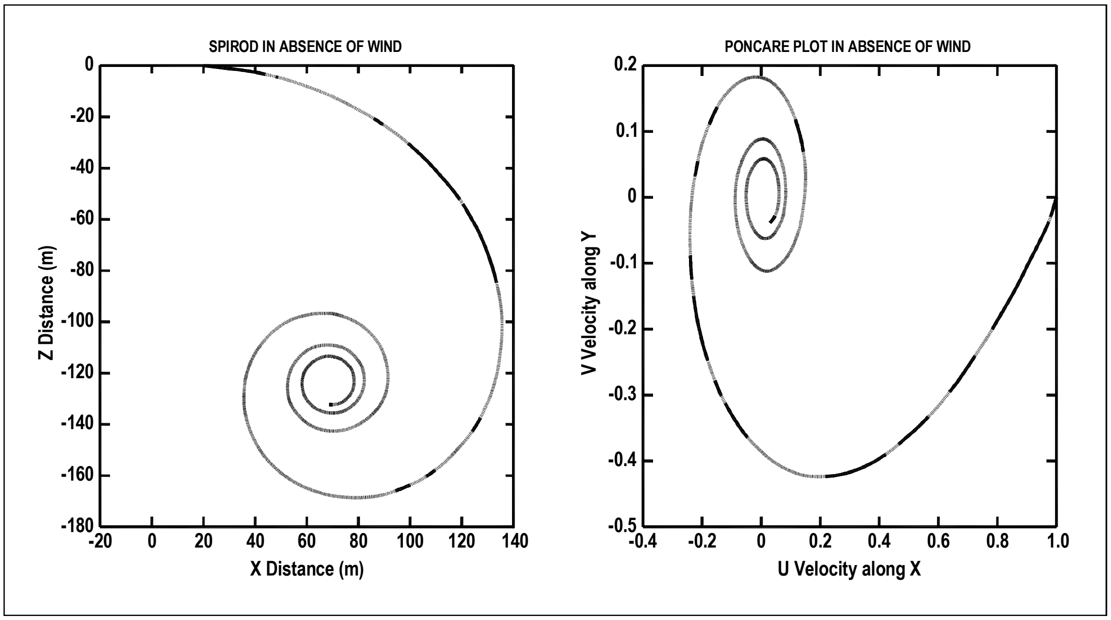


Figure 7. Spinning ball spiral in XZ-plane and velocity plot for $R_o = c_o$ and $s_2 = s_o$.

ball is thrown at speed $u_o = c_o$ with spin $s_2 = s_o$. These figures are similar to satellite pictures of cyclones.

8. Tail/head Wind Effect: Aerial View

There is a dramatic transformation in the flight path in the presence of a slight tail wind behind the spinning ball. The velocity diagram is shifted by c_w, but otherwise appears similar to *Figure* 7. The resulting flight path *Figure* 8, however, is drastically different. The tail wind takes the ball along a distorted cycloid. This path-called *spiroid*, blending spiral with the cycloid, displays exotic features unimaginable without spin. Depending on the magnitude of the tail wind, the flight path can exhibit both contracted and extended characteristics of a cycloid. This is clearly evident through the loops and cusps in *Figure* 8. Observe the final offset of the ball from the original direction of flight (which depends only weakly on wind speed). These results are at first counter intuitive, but it should be noted that we have used a large value of $s_2 = s_o$ to demonstrate these exotic variety of spinning ball trajectories. Two different magnitudes for the tail wind and one value for head wind illustrate

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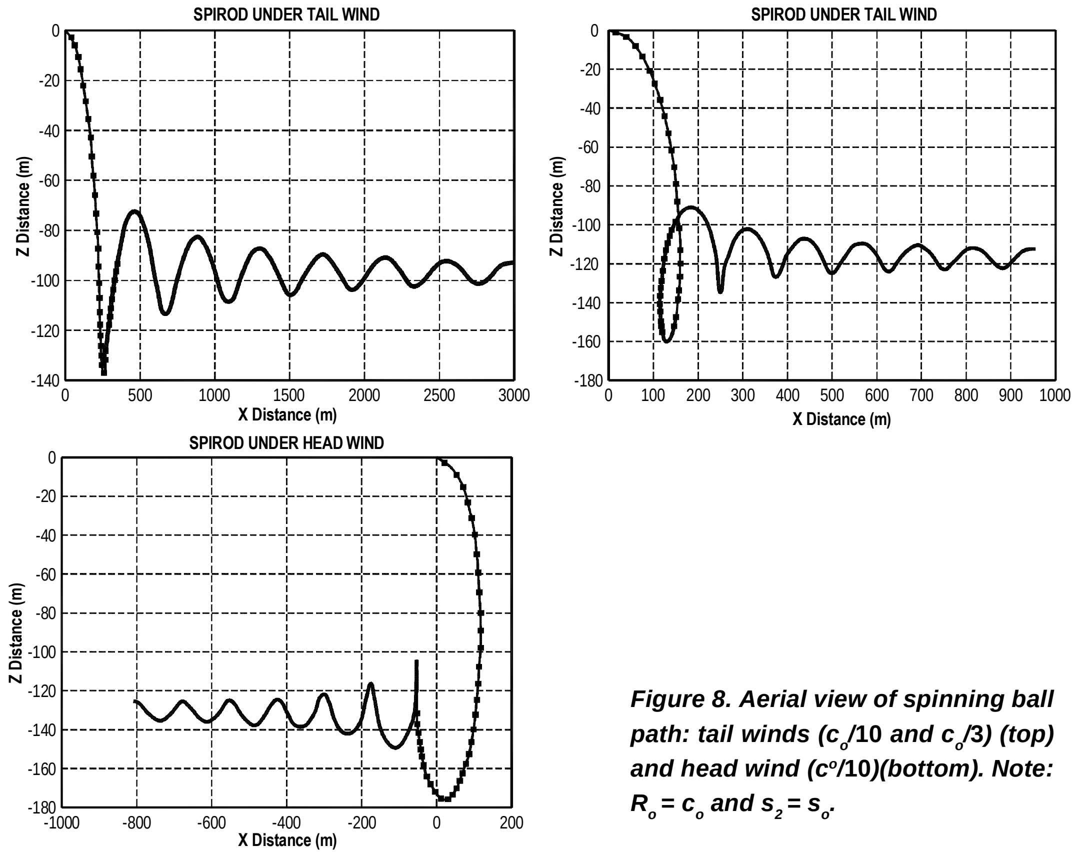

Figure 8. Aerial view of spinning ball path: tail winds (c_o/10 and c_o/3) (top) and head wind (c^o/10)(bottom). Note: $R_o = c_o$ and $s_2 = s_o$.

how the spinning ball snakes along until the tail/head winds dominate the dynamics.

In the context of tennis ball cricket or soccer, a player attempting to catch a spinning and snaking ball will have to be agile and alert. This effect also makes it possible for pitchers and bowlers to exploit wind and spin to confuse the batter. So, in tennis ball cricket, in addition to the leg-spin and off-spin, we have the snaking s-spin! These spiroidal paths are also observed during swirling dust storms and cyclones.

9. Summary

Though not just limited to soccer and cricket, spin is used by players to gain an advantage over their opponents. This makes the understanding of the dynamics of the spinning ball even more important as it could help players improve their game. The effects of spin are clearly seen in games like tennis and table tennis (TT). Here the player can impart 3 kinds of spins to the ball.

Forward spin or top spin: TT players will know how difficult it is to keep the ball on the small table when hitting a fast shot. This problem is overcome by imparting a top spin to ball while hitting a fast shot. The top spin causes the ball to dive, hence surprising the opponent and pitching before he expects it.

Backward spin: The backward spin imparted by a player counters gravity (the magnus effect) and this causes the ball to float in the air longer than expected. In this way even a slow shot can be made to cross the net by imparting a backward spin to the ball.

Side spin: Side spin is best observed during a serve in TT. When the player imparts a side spin to the ball the ball swings side ways (magnus effect) and moves further away from the opponent, making it tougher for him to receive the serve.

Thus, aerodynamic view of sports furnishes a deeper appreciation of the players skills and perseverance to master wind, spin and gravity. So far we discussed the dynamics of smooth spinning balls. But some balls (e.g., golf balls) have dimples on them, some are not circular (rugby balls); the dynamics of these balls are very different and much more challenging! The initial inspiration for this article came from observing the samara seeds spinning while falling down from the tree. Presently, there is a team of students designing a *paracopter* based on the spinning seed principle to achieve a terminal velocity of less than about 4 m/s for a 100 kg package dropped from a height of 100 m.

Suggested Reading

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