An Introduction to the Raychaudhuri Equations

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The Raychaudhuri equations are first introduced through simple examples and illustrations. Their use and the resulting consequences in cosmology are then briefly outlined.

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1. Preamble: Analogies and Illustrations

Equations, identities and inequalities provide a quantitative realisation of scientifically interesting phenomena. The level of generality of an equation (or identity, inequality) is usually understood by asking the question: how ubiquitous is it? Take for example Laplace's equation $\nabla^2\phi = 0$. It is well known that such an equation arises in many contexts – two of the most prominent ones being electricity and gravity.

In physics, if you look carefully, there aren't too many such ubiquitous equations. Those which are, have stood the test of time and have bloomed in many an unexpected scenario.

In 1955, while working on cosmology, Amal Kumar Raychaudhuri discovered one such equation. Today, this equation (which bears his name) has found its use in scenarios which he, probably never imagined. That is because, though cosmology is where it all began, the Raychaudhuri equations are, at a very basic level, largely geometric/mathematical statements. Hence, you can imagine applying them in many diverse situations which are far removed from the domains of cosmology.

Let us look at some of these situations now. The first of these is not an application in the true sense, but a nice way of introducing what the actual equations intend to tell us.

Keywords
Raychaudhuri equations, deformable media analysis, fluid flows, Einstein's general relativity, geodesics, cosmology.

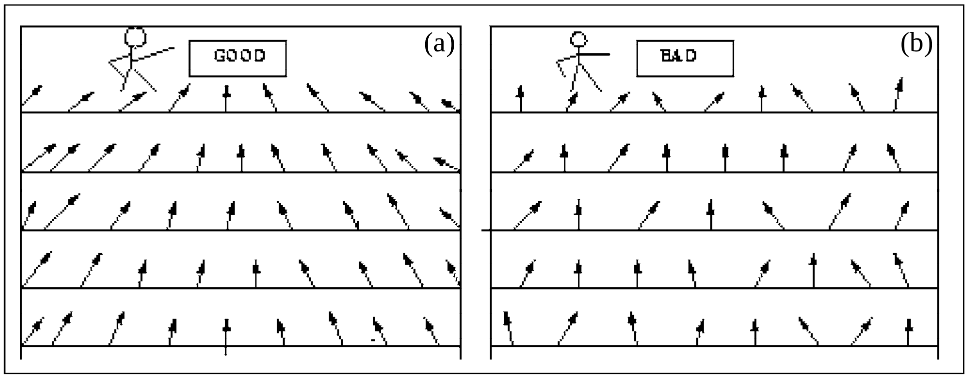

Figure 1. Vector field of look-at directions of the students' eyes for (a) a good lecture (b) a bad lecture.

a. *The Teacher-in-the-classroom Analogy*

Imagine a lecture hall (say, a really big one) full of students listening with rapt attention to an interesting talk. Think of little vectors which tell us in which *direction* a student is looking, at any moment of time. Map these vectors on separate sheets of paper (one for each instance of time during the talk), mark the location of the teacher and give it to a curious friend who wants to know if the talk was good or bad. At all instances, the friend finds that these arrows point toward a common location (near the blackboard)). He concludes that the talk must have been *interesting* – otherwise why would everyone keep looking at the blackboard continuously? In *Figure* 1 (a) and (b), we show the behaviour of this vector field of *look-at* directions.

In (a) we have the case when these arrows *focus* towards a point (a good teacher!).

In (b) they seem haphazardly distributed (the uninteresting or bad teacher!).

In summary, for an *interesting* lecture the look-at directions *focus* towards the blackboard.

b. *Deformable Media*

Our second example concerns deformable media. Suppose you are given a rubber disc. You are asked to deform it with an external force and then leave it alone, i.e., let the deformation evolve. Deforming it is of course easy, but how to quantify the way the deformation evolves in time? To quantify, you need to learn what are the types of deformations which may combine to give you a collective effect. Firstly, think of the situation when you give a radially inward/outward force along the circumference of the disc. The disc would then either isotropically expand or contract. Its shape will not change. On the other hand, you may give a force such that there is a shearing of the disc into an elliptical shape. Or, there could be a twist (a rotation) of the disc. In general, these are the only three essential ways in which you can deform the disc intially and leave it alone (let it evolve). Thus, the initial condition on the deformation may be an initial expansion, an initial shear or an initial rotation or a combination of all or some. *Figure* 2 illustrates the concepts of expansion, shear and rotation in this example.

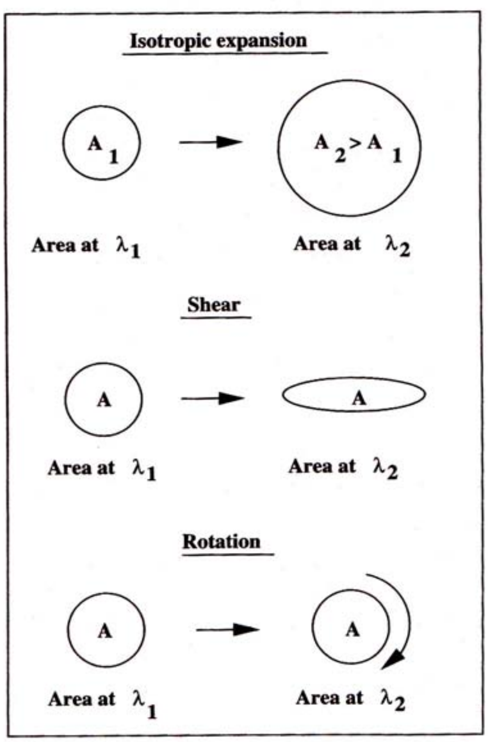

Figure 2. The defining features of expansion, shear and rotation of the disc as mentioned in the second example.

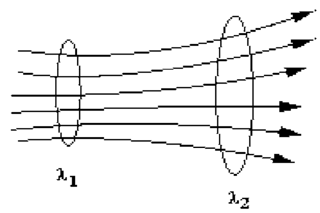

Figure 3. A set of fluid flow lines enclosing cross-sectional areas at two different values of the parameter λ. Note that the flow has a positive expansion: the area at λ_1 is smaller than that at λ_2.

c. *Flowing Fluids*

Finally, let us look at flowing fluids. Flowing fluids are represented by the velocity vector field defined at each point in the fluid. The velocity field is always tangential to the so-called fluid flow lines. Consider the cross-sectional area orthogonal to fluid flow at some value of a parameter (say λ) with which points in the flow lines are labelled. Take a fixed number of flow lines passing through this area and look at the same set of flow lines at another location (another value of λ) in the fluid. Does the area expand/contract, deform in shape or get twisted? If it does, then we have a situation similar to the case of deformable media. We can therefore characterise the flowing fluid by looking at how the cross-sectional area changes with the flow of the fluid. *Figure* 3 provides an illustration of the flow lines and the cross-sectional areas at two different values of λ. Differential equations governing the kinematic evolution of the expansion, shear and rotation can therefore tell us more about the nature of fluid flow. In other words, if you know the initial values of the expansion, shear and rotation, you can determine them at a later stage by solving these differential equations.

In all the three examples given above, we note a common feature – namely, the nature of evolution of a vector field along a parametrised direction. The parameter could be spatial (as in the first example) or it could be time (as in the second one) or an arbitrary parameter (as in the last example). The equations governing the evolution of the quantities such as expansion/contraction, shearing or twisting of such flow lines (as in the fluid case) are the Raychaudhuri equations (in a somewhat generalised sense). We shall make these ideas more precise below.

The equations governing the evolution of the expansion, shear or rotation along the flow lines are the Raychaudhuri equations.

2. Developing the Deformable Media Analogy

Let us now develop the second scenario given in the previous section, i.e., the case of deformable media. As we said before, we are not interested in asking how the deformation occured. That is to say – we are in the domain of kinematics. The obvious next question would be: What are the kinematical quantities? We have already learnt about the expansion, shear and rotation. We now define them a bit more precisely.

Consider a two-dimensional medium (example a sheet of rubber). Imagine that under some initial external stress the sheet is deformed at an arbitrary time $t = 0$. This stress produces the initial deformation which acts as a set of initial conditions. The stress is then removed and the system is allowed to evolve in time. What happens subsequently? To answer this question we need to solve the corresponding evolution equations for the deformation.

Denote the deformation by a vector ξ^i (where i = 1,2, i.e., we have a deformation vector with components ξ^1 and ξ^2 along the directions 1 and 2 respectively). The time rate of change of ξ^i can be analysed as follows: For small time intervals we may write

$$\xi^i(t_1) = \xi^i(t_0) + \Delta\xi^i(t_0), \tag{1}$$

where

$$\Delta\xi^i = B^i_j(t_0)\xi^j(t_0)\Delta t + \mathcal{O}((\Delta t)^2), \tag{2}$$

and the index j is summed over (eg. $\Delta\xi^1 = B^1_1\xi^1 + B^1_2\xi^2$. and, similarly for $\Delta\xi^2$). Therefore, we may write

$$\frac{\mathrm{d}\xi^i}{\mathrm{d}t} = B^i{}_j(t)\xi^j, \tag{3}$$

where $B^i{}_j(t)$ is an arbitrary second rank tensor and once again, the repeated indices (j, for instance, in the above equations) means we have to sum over them, eg. $\frac{\mathrm{d}\xi^1}{\mathrm{d}t} = B^1{}_1\xi^1 + B^1{}_2\xi^2$ and so on. If you are not familiar with tensors, think of the $B^i{}_j$ as a 2×2 matrix with elements

${B^1}_1(t)$, ${B^1}_2(t)$, ${B^2}_1(t)$ and ${B^2}_j(t)$ characterising the time evolution of the deformation vector.

Our scheme would now be as follows:

Step (i). Differentiate the above equation once more wrt time.

Step (ii). Convert it to an equation for ${B^i}_j(t)$ and its first time derivative with $\dddot{\xi}^i$ on its RHS.

Step (iii). Use the equation of motion for ξ^i as prescribed later and also equation (3) to arrive at the final result.

Implementing the Steps (i) and (ii) above (Step (iii) will be carried out later) we obtain:

$$\left(\frac{\mathrm{d}{B^i}_j}{\mathrm{d}t} + {B^i}_k {B^k}_j\right)\xi^j = \ddot{\xi}^i, \tag{4}$$

where, as before, the index k is summed over.

A. *Expansion, Rotation and Shear*

Before we write down the evolution equations let us first state the quantities of interest. The arbitrary second rank tensor ${B^i}_j$ can be decomposed into its trace, symmetric traceless and antisymmetric parts which will constitute the isotropic expansion (scalar), shear (symmetric traceless tensor) and rotation (antisymmetric tensor).

$$ {B^i}_j = \frac{1}{2}\theta\, {\delta^i}_j + {\sigma^i}_j + {\omega^i}_j. \tag{5}$$

Any arbitrary second rank tensor can be split into its trace, symmetric traceless and antisymmetric parts.

We can explicitly write the above expression in terms of the following 2×2 matrices:

$$\Theta \equiv \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix}, \Sigma \equiv \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix}, \Omega \equiv \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \tag{6}$$

where we have named the shear components as σ_+ and $\sigma_\times$ and the only rotation component as ω. Note that the matrix Θ has trace θ, the matrix Σ is traceless and symmetric, while the matrix Ω is antisymmetric. Thus, the four quantities θ, σ_+, $\sigma_\times$ and ω characterise any deformation of the two-dimensional deformable medium. The full $\mathbf{B}$ matrix with elements $B^i{}_j$ can be written down by just adding the above three matrices (i.e., matrix $\mathbf{B} = \Theta + \Sigma + \Omega$).

The Raychaudhuri equations constitute a coupled, first order, nonlinear system of ordinary differential equations.

***Exercise* 1.** Consider a circle of radius a. Any point on the circle can be parametrised as $(a\cos\phi, a\sin\phi)$. Find out how expansion, shear and rotation can deform the circular boundary. You have to find $\mathrm{d}\xi^i$ in each case – do it separately, once with the expansion matrix, then with the shear matrix and finally with the rotation matrix. Ignore order $(\Delta t)^2$ terms.

B. ***The Evolution (Raychaudhuri) Equations***

We now use the inputs from the previous subsection to rewrite the evolution equation for $B^i{}_j$ as four coupled nonlinear first order equations involving the dependent variables θ, σ_+, $\sigma_\times$ and ω. Before that, of course, we need to write down the equation of motion we assume for $\ddot{\xi}^i$. This, in a linear approximation, is taken as

$$\ddot{\xi}^i = -K^i_j\,\xi^j - \beta\,\dot{\xi}^i\,, \tag{7}$$

where β and $K^i{}_j$ denote damping and stiffness coefficients respectively and the matrix $\mathbf{K}$ (whose elements are the $K^i{}_j$) is assumed to be of the form:

$$\mathbf{K} \equiv \begin{pmatrix} k+k_+ & k_\times \\ k_\times & k-k_+ \end{pmatrix}. \tag{8}$$

The first term in (7) is nothing more than the well-known Hooke's Law[1]. It looks a bit different because we have included the possibility of anisotropic stresses ($\ddot{\xi}^1$ not just equal to $-K^1{}_1\xi^1 - \beta\dot{\xi}^1$ but, can be of the form $\ddot{\xi}^1 = -K^1{}_1\xi^1 - K^1{}_2\xi^2 - \beta\dot{\xi}^1$, where $K^1{}_2$ encodes the effect of anisotropy).

[1] Hooke's law of elasticity states that the amount by which a body is deformed is proportional to the force causing the deformation.

Finally, implementing Step (iii) mentioned earlier (i.e., the use of the equation of motion (7) for ξ^i and (3) in (4), and the elimination of ξ^i from both sides), the equations turn out to be:

$$\dot{\theta} + \frac{1}{2}\theta^2 + \beta\theta + 2k + 2(\sigma_+^2 + \sigma_\times^2 - \omega^2) = 0. \tag{9}$$

$$\dot{\sigma}_+ + (\beta + \theta)\,\sigma_+ + k_+ = 0. \tag{10}$$

$$\dot{\sigma}_\times + (\beta + \theta)\,\sigma_\times + k_\times = 0. \tag{11}$$

$$\dot{\omega} + (\beta + \theta)\,\omega = 0. \tag{12}$$

We note that the equations for σ_+, $\sigma_\times$ and ω are structurally similar whereas the equation for θ is known in the literature as a Riccati differential equation. Further, the equations constitute a coupled, first order, nonlinear system.

***Exercise* 2.** Obtain equations (9)–(12) by using equations (3), (4), (7) and (8). The way to do it is to rewrite the equation (4) using equations (7) and (3) first. This will turn out to be $\frac{\mathrm{d}B^i{}_j}{\mathrm{d}t} + B^i{}_k B^k{}_j = -K^i{}_j - \beta B^i{}_j$ after eliminating ξ^i from both sides. Then, explicitly write down equations for $\frac{\mathrm{d}B^1{}_1}{\mathrm{d}t}$, $\frac{\mathrm{d}B^1{}_2}{\mathrm{d}t}$, $\frac{\mathrm{d}B^2{}_1}{\mathrm{d}t}$ and $\frac{\mathrm{d}B^2{}_2}{\mathrm{d}t}$. Finally, use equation (6) and the combinations $\frac{\mathrm{d}}{\mathrm{d}t}(B^1{}_1 \pm B^2{}_2)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(B^1{}_2 \pm B^2{}_1)$ to get the four equations (9)–(12).

Solving the equations (9)–(12), one can determine how the quantities θ, σ_+, $\sigma_\times$ and ω evolve in time. Though complicated, the equations can indeed be solved. A generalisation to three dimensions can also be worked out. The interested reader can look up the second reference for further details.

***Exercise* 3.** Assume k_+ and $k_\times$ as zero. Also take $\beta = 0$. Show that if one defines a quantity $I = \sigma_+^2 + \sigma_\times^2 - \omega^2$, then the four equations can be recast into two equations involving I and θ. Try to solve these equations and obtain the evolution of the expansion, shear and rotation.

3. Fluid Flows

As mentioned in the first section, the kinematics of fluid flow can be understood using a similar formalism. In order to make contact with cosmology we now bring in the relativistic formalism. The things needed to know about such a formalism are listed in *Box* 1.

Comparing with the case of deformable media, we notice that the deformation vector ξ^i is now replaced by the velocity vector v^i, where i here runs from 0 to 3. Also, we assume v^i to be time-like (see *Box* 1 for the definition of time-like).' We consider only those flow lines which are geodesic, in the sense that v^i are tangents to geodesics. The gradient of the velocity vector is of course $\frac{\partial v^i}{\partial x^j}$. When we have a curved space-time (recall what Einstein taught us in General Relativity: the gravitational field is equivalent to a curved space-time, see *Box* 1 for further details), the usual partial derivative is replaced by a covariant derivative, denoted by ∇_j. Thus, in summary, the quantity $\nabla_j v_i$ is a second rank tensor and can be split, as before into the trace, symmetric traceless and antisymmetric parts. These are once again, the expansion, shear and rotation.

The behaviour of the expansion along the flow would tell us whether the flow lines come towards each other or move away.

So, the equations of evolution of the expansion, shear and rotation along the flow lines can easily be obtained by further differentiating $\nabla_j v_i$, using the fact that the trajectories are geodesic and then extracting the trace, symmetric traceless and antisymmetric parts on both sides (recall, these are exactly the same as Steps (i), (ii) and (iii) mentioned earlier, for deformable media). This is a bit complicated to work through, so we mention what the end result is and then discuss the consequences. We look at only the equation for the expansion θ, because the other equations are more complicated and also because Raychaudhuri, in his original paper, did choose to discuss this equation in greater detail.

Box 1.

Space-time: In special relativity, time t is elevated to the status of another coordinate. $txyz$ constitutes a four-dimensional space-time continuum. One talks about space-time and not space and time. The infinitesimal distance between two points in space-time is given as $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ (Minkowski distance), which can be positive (space-like), negative (time-like) or zero (null). This is unlike the usual distance we are familiar with, i.e., $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ which is always positive.

Transformations: To go from one inertial frame $txyz$ to another $t'x'y'z'$, one has to use the Lorentz transformations and *not* the Galilean transformations. The Minkowski distance mentioned above remains unchanged under a Lorentz transformation of the coordinates.

Four vector: Like space-time co-ordinates, vectors also have a fourth component, hence the name four vector. For instance, the velocity four vector now has a fourth component which is nothing but γc (with $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$). Four vectors will transform via Lorentz transformations and can have length as positive (space-like), negative (time-like) or zero (null).

Curved space-time: General Relativity goes further in order to incorporate gravity via the Equivalence Principle. Locally inertial frames (locally like those of Special Relativity) encode the global effects of gravity through a nontrivial, curved (as opposed to flat, which is the case with the geometry defined with the Minkowski distance mentioned at the beginning of this box) space-time geometry. Between two such locally inertial frames one transforms using completely general coordinate transformations.

Geodesics: Geodesics are trajectories of test particles in a given gravitational field. Since such fields are equivalent to a curved space-time, the trajectories are those of extremum distance in a given metric geometry (geometry is specified through the distance function, e.g., on a two-dimensional sphere of radius a, the distance (infinitesimal) is $\mathrm{d}s^2 = a^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) = g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2 \rightarrow$ the metric functions, $g_{\theta\theta} = a^2$ and $g_{\phi\phi} = a^2\sin^2\theta$). The tangent to a geodesic curve, at a specific point on it, is a four vector.

Curvature: A measure of the curvature of a geometry is the Riemann curvature, which involves second derivatives of the metric functions. Among derived quantities is the Ricci tensor (the R_{ij} is the Ricci tensor which appears in the Raychaudhuri equation). Einstein's equations of gravity (the equations which replace the Newtonian equation $\nabla^2\phi = 4\pi G\rho_m$) may be written non-technically as $Geometry = Matter$, where the geometry part involves quantities constructed out of the Ricci tensor and the metric functions.

The equation for the expansion is indeed rather simple. We have already written a similar equation (look at equation (9) with $\beta = 0$) while discussing about deformable media. Here it is again:

$$\frac{d\theta}{d\lambda} + \frac{1}{3}\theta^2 + \sigma^2 - \omega^2 = -R_{ij}v^i v^j, \qquad (13)$$

where $\sigma^2 = \sigma_{ij}\sigma^{ij}$ and $\omega^2 = \omega_{ij}\omega^{ij}$. The quantity that appears on the RHS is, essentially, 'geometric' (or, equivalently following Einstein, is related to the 'matter', which curves geometry). Note that this term can be thought of as the parallel of $2k$ in equation (9). Thus, just as k in equation (9) can be a function of t (time-dependent spring constant), the $R_{ij}v^i v^j$ term is, in general, a function of the parameter λ. The expansion, following the fluid flow scenario, would tell us whether the flow lines come together, or go apart from each other. λ is a parameter that labels points on the flow lines. So, if the flow lines come closer, the question is 'what is causing this to happen?'. There is nothing else which can do it other that the term on the RHS and the third and fourth terms on the LHS in equation (13). If, for the time being we set the rotation to zero, we will show below that it is the attractive nature of gravity which causes the lines to focus. So, going back to the first analogy, we might say that the idea of a 'good teacher' is replaced by 'gravity' and the fact 'the students eyes focus to the blackboard' is replaced by 'fluid flow lines focus towards a point'. This is what Raychaudhuri taught us – indeed a very simple fact – that gravity, by virtue of being attractive, causes geodesics (flow lines) to focus towards a point. Let us now see whether we can make this a little more quantitative, by analysing the equation (13).

The good teacher/ talk is like gravity and the fact that students' eyes focus towards the board is like focusing of trajectories.

As we said before, we will set the rotation term to zero. We also make a transformation

$$\theta = \frac{3}{F}\frac{dF}{d\lambda}. \qquad (14)$$

The divergence of an initially negative θ to negative infinity within a finite value of λ is focusing.

With this transformation the equation becomes

$$\frac{\mathrm{d}^2F}{\mathrm{d}\lambda^2} + \frac{1}{3}\left[R_{ij}v^iv^j + \sigma^2\right]F = 0. \tag{15}$$

The contents in the square brackets can be termed as some function of λ. Note that the velocity vector is obviously a function of λ and the elements of R_{ij} are the functions of the coordinates which are, in turn, functions of λ.

So, the equations resemble something very familiar to all of us – a harmonic oscillator equation with a time dependent spring constant:

$$\frac{\mathrm{d}^2F}{\mathrm{d}\lambda^2} + k(\lambda)F = 0. \tag{16}$$

However, it is a harmonic oscillator type equation only if k is a positive quantity. This means $R_{ij}v^iv^j + \sigma^2 \geq 0$. Or, with $\sigma^2 \geq 0$, we have the requirement $R_{ij}v^iv^j \geq 0$.

Now, one may ask – what do we get out of this harmonic oscillator-like equation? To understand this, we need to go back to the definition of $\theta = \frac{3}{F}\frac{\mathrm{d}F}{\mathrm{d}\lambda}$. If F is oscillatory, then F will have zeros at finite values of λ, (remember, functions like the sine or cosine have zeros at finite values of λ). This would imply that θ can diverge to negative infinity if, initially (at say, $\lambda = 0$), θ is negative. Now recall that θ is a quantifier of the isotropic expansion. Therefore we may write θ as

$$\theta = \frac{A(\lambda_2) - A(\lambda_1)}{A(\lambda_2)}, \tag{17}$$

where $A(\lambda)$ is the cross-sectional area enclosing a fixed number of flow lines. Thus, if $A(\lambda_2)$ goes to zero (i.e., the congruence collapses to a point), then θ goes to negative infinity. This is geodesic focusing and it happens because gravity is attractive. The condition $R_{ij}v^iv^j \geq 0$ can be shown, using Einstein's equations of General

Relativity as related to 'gravitating matter', which is attractive in nature. Loosely speaking, $R_{ij}v^iv^j$ is somewhat like $\nabla^2\phi$ (with ϕ as the gravitational potential). Therefore, using Poisson's equation for Newtonian gravity, $\nabla^2\phi$ is proportional to the mass density ρ_m, which as we all know, is always ≥ 0.

***Exercise* 4.** Assuming $\omega_{ij} = 0$ and $R_{ij}v^iv^j \geq 0$, one can recast the Raychaudhuri equation as an inequality $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} + \frac{1}{3}\theta^2 \leq 0$. Analyse this inequality to show that if θ at λ_0 is negative, then θ must tend to negative infinity within a finite value of λ.

You might be curious, at this stage, as to why we are always setting the rotation equal to zero? If you don't, you will get an opposite effect, because the sign of that term in equation (13) is opposite to that of the σ^2 and the $R_{ij}v^iv^j$ terms. Thus rotation can actually lead to a *defocusing* (geodesics moving away from each other) effect.

4. Cosmology

As mentioned right at the beginning, the equations originally arose in the context of cosmology. In a model of the universe which we call a cosmological model, the universe is taken as homogeneous and isotropic (i.e., no preferred locations or directions) at scales in which a galaxy is treated as a point. Such galaxies make up the cosmological fluid which flows along geodesics in the given geometry. The expansion of the universe is quantified by a scale factor which we denote as $a(t)$. The $a(t)$ can be obtained by solving the Einstein equations with an appropriate source, usually a perfect fluid. So, it can be seen, how fluid flow enters the picture. Obviously, once we have a flow, we must have expansion, shear and rotation. It turns out that the expansion is given quite simply:

$$\theta = 3\frac{\dot{a}}{a} = \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}a^3. \tag{18}$$

Box 2.

What is a *singularity*? Loosely speaking, it is a location where 'things' are ill-defined. Recall Coulomb's law or Newton's law of gravity and ask yourself – what happens when you are just at the location of the point charge or the point mass – the potential and the field diverges to infinity. Einstein taught us that gravity is manifest in the curvature of space-time – so when the curvature becomes very large, gravity must be very strong too. More specifically, when certain co-ordinate invariant quantities (scalars) diverge at some point, we say that we have a *curvature singularity*. The Big Bang and the black hole are two examples, though they are different in nature. For instance, in the case of Big Bang, the scalars just mentioned about, vary as inverse powers of the scale factor $a(t)$ – so when $a(t) \to 0$ (the universe shrunk to a point in its past, as it is believed to have been) these quantities diverge. However, note there is another characteristic of a singularity – geodesics (trajectories) seem to end there, they cannot be continued further beyond and this happens at a finite value of the parameter that labels points on them. Such situations are described technically by the term *geodesic incompleteness*. But it might happen that geodesic incompleteness can occur without there being any curvature singularity, i.e., those scalars may not diverge at that point beyond which geodesics cannot be continued. So, its a *one-way* statement which we can make – curvature singularities are locations where curvature diverges and geodesics end (focus) but geodesics can focus in a completely benign way too (without encountering a curvature singularity). In a nutshell, curvature singularities imply geodesics focusing there, but, geodesic focusing does not necessarily imply a curvature singularity. Thus, you realise how important the Raychaudhuri equations must be in understanding these singularities, which, despite Einstein's utter dislike of them, are inevitably there in his theory of gravity. The details about singularities (their definition, properties, the conditions under which they are bound to appear etc.,) is what Hawking and Penrose worked on in the 1960s, early 1970s and one of their tools (among many others) were the Raychaudhuri equations.

If we take $a(t) \sim t^{\nu}$, then $\theta \sim t^{-1}$. Going backwards in time (i.e., towards $t \to 0$ from today), it is therefore easy to see that we end up with an expansion which becomes negative infinity. That's *focusing* – and where do these geodesics focus to? – the big bang! So, inevitably, the existence of a singularity (see *Box* 2 to get an idea of a singularity) becomes obvious irrespective of what ν is. Further, looking at the second expression for θ (i.e., $\theta = \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}a^3$), one can easily conclude that θ is a quantity which characterises the fractional change in the volume of the three-dimensional space (volume $= a^3$). Therefore, it is also called the *volume* expansion. In fact the Raychaudhuri equation and geodesic focusing gave the first hint that singularities are inevitable in Einstein's General Relativity. Later on in the late 1960s and early 1970s, Hawking and Penrose proved the so-called singularity theorems on the basis of Raychaudhuri equations!

Exercise 5: If you know how to calculate the Ricci tensor R_{ij} then, assuming shear and rotation as zero and $v^i \equiv (1, 0, 0, 0)$, show that the θ given in equation (18) satisfies the Raychaudhuri equation for the expansion.

Suggested Reading

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We couldn't wait for the apple to drop...