Snippets of Physics

4. Schwarzschild Metric at a Discounted Price[*]

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The gravitational field of a massive, spherical, body like the Sun is described in general relativity by a solution to Einstein's equations called the Schwarzschild soution. Here is an elementary persepective on this solution, which – though far from a rigourous derivation – raises intriguing questions.

The correct description of gravity is based on Einstein's general theory of relativity which, unfortunately, has a reputation of being difficult to learn. How would you like to get a key result of general relativity rather cheaply? In this installment, I will discuss how an important result of general relativity, viz., the description of the gravitational field of a spherical body, can be obtained using only the concepts of special relativity. This curious fact allows you to explore a host of physical phenomena including some aspects of blackhole physics. However, like some of the stunts which appear on TV, the derivation given here should come with the warning: "This stunt is performed by experts and do not try this on your own at home"! The derivation works – for reasons which are not completely clear – only for a special class of spherically symmetric models but considering how easy it is, it deserves to be known much better.

Let us begin with a simple idea which you probably know. Gravity obeys the *principle of equivalence*. Consider, for example, a small box ('Einstein's elevator'[1]) which is moving in intergalactic space, away from all material bodies, in some direction ('up') with a uniform acceleration g. We will assume that it is propelled by the rocket motors attached to its bottom. Let us compare

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[1] Einstein's elevator is a box which is accelerating upwards in intergalactic space, far away from all material bodies. Einstein used this concept very effectively to do his thought experiments!

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the results of any physical experiment performed inside such an elevator with those performed inside a similar box which is at rest on Earth's surface where the gravitational acceleration is g. The principle of equivalence tells us that the results of all physical experiments will be the same in these two cases; that is, one cannot distinguish gravity from an accelerated frame within any sufficiently small region.

If you jump off from the twentieth floor of a building you will feel completely weightless ('zero gravity') until you crash to the ground (though this is not the recommended procedure to verify the principle of equivalence!).

An immediate consequence of this principle is that you can make gravity go away, within any small region of space, by choosing a suitable frame of reference usually called freely falling frame. For example, if you jump off from the twentieth floor of a building you will feel completely weightless ('zero gravity') until you crash to the ground (though this is not the recommended procedure to verify the principle of equivalence!). In such a freely falling frame, one can use the laws of special relativity without any problem since gravity is absent.

We want to use this idea to describe the gravitational field of a spherically symmetric body located about the origin. The body has a radius say, R, and we are studying the gravitational field in the empty space at $r > R$. Let P be a point at a distance r from the origin. If we consider a small box around P which is freely falling towards the origin, then the metric in the coordinates used by a freely falling observer in the box will be just that of special relativity:

$$\mathrm{d}s^2 = c^2\mathrm{d}t_{\mathrm{in}}^2 - \mathrm{d}\mathbf{r}_{\mathrm{in}}^2. \quad (1)$$

This is because, in the freely falling frame, the observer is weightless and there is no effective gravity (*Figure* 1). (The subscript 'in' is for inertial frame.) Let us now transform the coordinates from the inertial frame to a frame $(T, \mathbf{r})$, which will be used by observers who are at rest around the point P. Suppose the freely falling frame is moving with a radial velocity $\mathbf{v}(r)$ around P. To determine this velocity, we can imagine that the

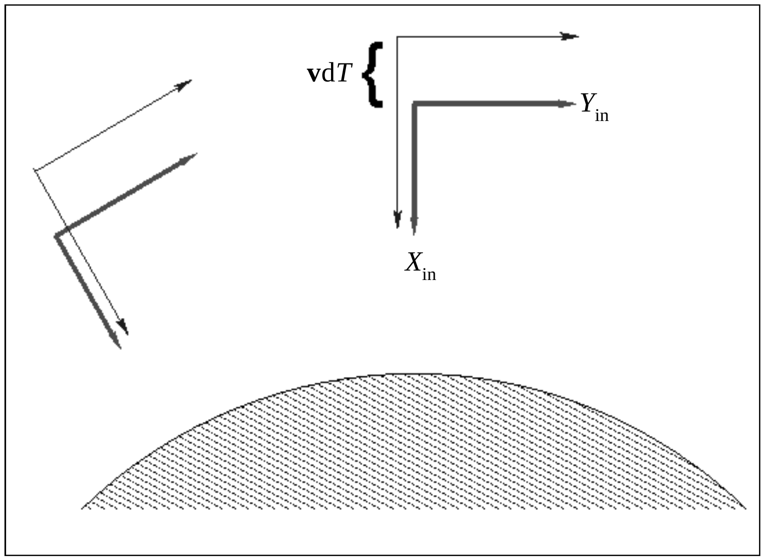

Figure 1. The relation between freely falling and static coordinate frames around a spherically symmetric body. The thick red lines indicate (X_{in},Y_{in}) axes of the freely falling inertial frame. The blue thin lines denote the corresponding axes of a static coordinate system glued at a fixed point. (Of course, the figure is not to scale and the coordinates are supposed to be infinitesimal in extent!). The radial displacement between the two frames is by the amount $v_r(r)$ dT during an infinitesimal time interval dT. Through every event there is a different freely falling frame related in a definite way to the fixed static frame. The freely falling and static frames coincide at very large distance from the body.

freely falling frame started from very large distance from the body with zero velocity at infinity. Then, a simple Newtonian analysis shows that its velocity at P will be $\mathbf{v}(r) = -\hat{\mathbf{r}}\sqrt{2GM/r}$. We now transform from the freely falling inertial frame to the static frame of reference which is glued to the point P using the non-relativistic (Galilean) transformations $\mathrm{d}t_{\text{in}} = \mathrm{d}T, \mathrm{d}\mathbf{r}_{\text{in}} = \mathrm{d}\mathbf{r} - \mathbf{v}\mathrm{d}T$ between two frames which move with respect to each other with a relative velocity $\mathbf{v}$. Of course, you have to use infinitesimal quantities in this transformation because you need different freely falling inertial frames at different points, in a nonuniform gravitational field.

What we really want is the form of the line element in equation (1) in terms of the static coordinates. Substituting the transformations $\mathrm{d}t_{\text{in}} = \mathrm{d}T, \mathrm{d}\mathbf{r}_{\text{in}} = \mathrm{d}\mathbf{r} - \mathbf{v}\mathrm{d}T$ in equation (1), we find the metric in the new coordinates to be

$$\mathrm{d}s^2 = \left[1 - \frac{2GM}{c^2 r}\right] c^2 \mathrm{d}T^2 - 2\sqrt{(2GM/r)}\mathrm{d}r\mathrm{d}T - \mathrm{d}\mathbf{r}^2. \tag{2}$$

Incredibly enough, this turns out to be the correct metric *describing the space-time around a spherically symmetric mass distribution of total mass M!* As it stands, this line element is not in 'diagonal' form in the sense that it has a non-zero $\mathrm{d}r\mathrm{d}T$ term. It will be nicer to have the metric in diagonal form. This can be done by making a coordinate transformation of the time coordinate (from T to t) in order to eliminate the off-diagonal term. We look for a transformation of the form $T = t + J(r)$ with some function $J(r)$. This is equivalent to taking $\mathrm{d}T = \mathrm{d}t + K(r)\mathrm{d}r$ with $K = dJ/\mathrm{d}r$. Substituting for $\mathrm{d}T$ in equation (2), we find that the off-diagonal term is eliminated if we choose $K(r) = \sqrt{2GM/c^4 r}(1-2GM/c^2 r)^{-1}$. (I will leave the straight forward algebra for you to work out.) In this case, the new time coordinate is:

Schwarzschild metric is used extensively both in the study of general relativistic corrections to motion in the solar system and in black hole physics.

$$t = \int \mathrm{d}T + \frac{1}{c^2}\int \mathrm{d}r \frac{\sqrt{(2GM/r)}}{(1-\frac{2GM}{c^2 r})}. \quad (3)$$

The integral in the second term is elementary and working it out you will find that the transformation we need is:

$$ct = cT - \left[\sqrt{\frac{8GMr}{c^2}} - \frac{4GM}{c^2}\tanh^{-1}\sqrt{\frac{2GM}{c^2 r}}\right]. \quad (4)$$

What is more important for us is the final form of the line interval in equation (2) expressed in the static coordinates with the new time coordinate t. This is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \quad (5)$$

Some of you might have seen this metric, called the *Schwarzschild metric*, which is used extensively both in the study of general relativistic corrections to motion in the solar system and in black hole physics.

A clock located in a gravitational potential well of a massive body runs slowly compared to a clock located at large distances from the body.

Once we have the form of this metric, we can do lots of things with it using just special relativistic concepts. One simple but very significant result which you can obtain immediately is the following: Let us consider a clock which is sitting quietly at some fixed location in space so that, along the clock's world line, $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\phi = 0$. Substituting these into equation (5), we get the proper time shown by such a clock to be

$$\mathrm{d}\tau = \left[1 - \frac{2GM}{c^2 r}\right]^{1/2} \mathrm{d}t \equiv \sqrt{g_{00}(r)}\mathrm{d}t, \tag{6}$$

when the coordinate clock time changes by an amount $\mathrm{d}t$. It is obvious from this relation that $\mathrm{d}\tau \to \mathrm{d}t$ when $r \to \infty$. That is, one can think of t as the proper time measured by a clock located far away from the gravitating body. The result shows that a clock located in a gravitational potential well of a massive body runs slowly compared to a clock located at large distances from the body.

Consider now an electromagnetic wave train made of N crests and troughs which is travelling radially outward from some point $\mathbf{x}$ to an infinite distance away from the central body. An observer located near $\mathbf{x}$ can measure the frequency of the wave train by measuring the time $\Delta\tau$ it takes for the N troughs to cross her and using the result $\omega = N/\Delta\tau$. An observer at large distances will do the same using her clock. Since the frequency of radiation $\omega(\mathbf{x})$ measured by local observers, as the radiation propagates from event to event in a curved spacetime, is inversely related to the time measured by the local clock, it follows from equation (6) that $\omega(\mathbf{x}) \propto g_{00}(\mathbf{x})^{-1/2}$. If $g_{00} \approx 1$ at very large distances from a mass distribution, then the frequency of radiation measured by an observer at infinity (ω_∞) will be related to the frequency of radiation emitted at some point $\mathbf{x}$ by $\omega_\infty = \omega(\mathbf{x})\sqrt{g_{00}(\mathbf{x})}$. This effect is called gravitational redshift.

You can now investigate all kinds of physics around a spherical gravitating body, in general relativity, starting from these concepts. As an illustration, we will briefly discuss how to study orbits of particles in general relativity. To obtain this result in the simplest manner, let us begin with the trajectory of a particle in special relativity under the action of a central force. The angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved for the particle but the momentum is now given by $\mathbf{p} = \gamma m \mathbf{v}$ with $\gamma \equiv (1 - v^2/c^2)^{-1/2}$. So the relevant conserved component of the angular momentum is $L = mr^2(\mathrm{d}\theta/\mathrm{d}\tau) = \gamma mr^2(\mathrm{d}\theta/\mathrm{d}t)$ rather than $mr^2(\mathrm{d}\theta/\mathrm{d}t)$. (This, incidentally, means that Kepler's second law regarding areal velocity does not hold in special relativistic motion in a central force if we use the coordinate time. The particle will cover equal areas during equal lapses of proper time.) Consider now the motion of a *free* special relativistic particle described in *polar* coordinates. The standard relation $E^2 = \mathbf{p}^2c^2 + m^2c^4$ can be manipulated to give the equation

Kepler's second law regarding areal velocity does not hold in special relativistic motion in a central force if we use the coordinate time.

$$\frac{E^2}{c^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = E^2 - \left(\frac{L^2c^2}{r^2} + m^2c^4\right). \qquad (7)$$

(This is still the description of a *free* particle moving in a straight line but in the polar coordinates!) Since special relativity must hold around any event, we can obtain the corresponding equation for general relativistic motion by simply replacing $\mathrm{d}r$, $\mathrm{d}t$ by the proper quantities $\sqrt{|g_{11}|}\mathrm{d}r$, $\sqrt{|g_{00}|}\mathrm{d}t$ and the energy E by $E/\sqrt{|g_{00}|}$ (which is just the redshift obtained above) in this equation. This gives the equation for the orbit of a particle of mass m, energy E and angular momentum L around a body of mass M. With some simple manipulation, this can be written in a suggestive form as:

$$\left(1 - \frac{2GM}{c^2r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{E}\left[E^2 - V_{\mathrm{eff}}^2(r)\right]^{1/2} \qquad (8)$$

with an effective potential:

$$V_{\text{eff}}^2(r) = m^2c^4\left(1 - \frac{2GM}{c^2r}\right)\left(1 + \frac{L^2}{m^2r^2c^2}\right). \quad (9)$$

You can now work out various features of general relativistic orbits exactly as you do it in standard Kepler problem. And the above derivation clearly shows that the particle is essentially following special relativistic, *free particle* motion at any event, in the locally inertial coordinates! This is really general relativity for the price of special relativity.

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Suggested Reading

[1] The derivation of Schwarzschild metric using the freely falling frame is discussed in detail in M Visser, 2003, *Heuristic approach to the Schwarzschild geometry*; gr-qc/0309072.
For a textbook discussion, see T Padmanabhan, *An Invitation to Astrophysics*, World Scientific, 2006.

[2] The effective potential as well as orbits in Schwarzschild metric are discussed in any standard text in general relativity, C W Misner, K S Thorne and J A Wheeler, *Gravitation*, Freeman, ch.25, 1973.