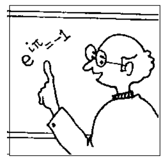

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Why Did Veeru Always and Inzamam Often Lose the Toss?

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In this note we describe a method of conducting a fair toss using any given coin, fair or otherwise. We also consider some generalizations of the method described.

Conducting a Toss

Many people of my generation, who 'grew up' on Amitabh Bachchan's movies, must have watched the blockbuster 'Sholay', which has since achieved a cult status, many times over. Apart from the pulsating drama and unforgettable dialogues, the movie had many spellbinding features. Of particular interest, in the context of the present article is the one involving the two protagonists Veeru and Jai, who were small time criminals. They would resort to tossing a coin whenever they had very selfless differences. The coin belonged to Jai and he would always toss and call heads. Poor unsuspecting Veeru, who lost every single time the coin was tossed, discovered, much to his mortification, only after the death of his beloved friend Jai, that the coin actually had heads on both sides.

Keywords.
Coin tossing, fair toss, geometric distribution, equal and varying probability selection.

There is a widespread perception among the critics as well as followers of the game of cricket, that some captains like Inzamam-ul-Haq, more often than not, lost the toss.

So did Jai use a fair coin? Your answer of course, would be a loud NO. Needless to say that in the case of Jai and Veeru, who were bosom pals, the actions were driven by selfless considerations of mutual well-being and the coin was heavily loaded in favour of Jai. Reality, however, may not always be as romantic or altruistic as fables or movies. There are occasions when we need to toss a coin, especially in sporting events. Prior to the commencement of any cricket match we see the two captains, umpires and a commentator come together for the 'toss'. Even in many other games like soccer, hockey or tennis, any match is preceded by a toss. In cricket the winner decides whether to bat or field first. In service games, the winner chooses to serve or opts for either of the half courts and so on.

There is a widespread perception among the critics as well as followers of the game of cricket, that some captains like Inzamam-ul-Haq, more often than not, lost the toss. Is there any substance to such a perception? Can it be validated? Or are the tosses always 'fair'? In all these situations, the underlying assumption is that the coin that is being tossed is a fair one. What if such an assumption is not correct? Is that why Inzamam often lost the toss? If so, how do we remedy that? In what follows we try to answer these questions.

Suppose the coin at our disposal, when tossed, turns up heads with probability θ, $\theta \in (0,1)$. The frequentist's interpretation of θ is if we toss a given coin for a sufficiently large number of times, the proportion of number of heads observed to number of tosses is approximately θ. When $\theta = \frac{1}{2}$, we say that we have a fair coin. As far as any single toss is concerned, it may be said that a fair coin when tossed is equally likely to turn up heads or tails. Obviously for any sporting event it is not only desirable but just as well to use a fair coin.

Are the tosses always 'fair'? For any sporting event it is not only desirable but just as well to use a fair coin.

At this stage one may think of two distinct possibilities for the coin at our disposal:

a) θ may be different from $\frac{1}{2}$, say $\frac{2}{5}$, for example, or worse

b) θ may not even be known. In either case how do we conduct a 'fair toss' using the given coin?

Let H denote the outcome that the coin when tossed turns up heads and T denote the outcome that the coin turns up tails. Suppose we perform a trial that consists of two (independent) tosses of the given coin. Now there are four outcomes of any trial, namely, HH, HT, TH and TT. Let us say, Rahul Dravid and Inzamam-ul-Haq are the rival captains. The captain who calls is allowed to call either HT or TH. If Inzamam is the calling captain and he calls, say, HT and if HT is the outcome then he wins, if TH is the outcome then the rival captain Rahul wins and if the outcome is either HH or TT, then the trial is rendered ineffective and another trial is performed. This process is continued until such times as either of the captains wins.

A natural question that comes to one's mind at this stage is what is the guarantee that such an elaborate procedure will terminate? And if it indeed does, then does it ensure a 'fair toss' i.e., whether both Inzamam and Rahul have the same chance of winning? In what follows we try to answer these questions.

As mentioned before, let the given coin, when tossed, turn up heads with probability $\theta, \theta \in (0,1)$. Further the coin is tossed independently. Under these assumptions the chances of the four possible outcomes are as follows:

Outcome	HH	HT	TH	TT
Probability	θ^2	$\theta(1-\theta)$	$(1-\theta)\theta$	$(1-\theta)^2$

Thus the outcomes HT and TH have the same chance of occurring, namely, $\theta(1-\theta)$.

Let η be the probability that a trial is ineffective. Thus η is same as the probability that the outcome is either *HH* or *TT*.

We say that a trial is ineffective if the outcome is either HH or TT. Equivalently, we say that a trial is effective if the outcome is either HT or TH. Let η be the probability that a trial is ineffective. Thus η is same as the probability that the outcome is either HH or TT. Therefore $\eta = \theta^2 + (1-\theta)^2$ or $\eta = 1 - 2\theta(1-\theta)$.

We first find the probability that the procedure terminates. Clearly the procedure will terminate as soon as we have an effective trial. Thus, the procedure terminates if the first trial is effective or the first trial is ineffective, and the second trial is effective or the first two trials are ineffective, and the third trial is effective and so on.

Thus the probability that the procedure terminates is given by

$$(1-\eta) + \eta(1-\eta) + \eta^2(1-\eta) + \eta^3(1-\eta) + \cdots$$
$$= (1-\eta)\left(1 + \eta + \eta^2 + \eta^3 + \cdots\right) = (1-\eta)\frac{1}{(1-\eta)} = 1.$$

We may take heart from this observation that ours is not an unending procedure and it indeed terminates with probability 1.

We now turn to finding the probability of Inzamam winning.

Note that Inzamam wins if the first trial results in HT, or the first trial is ineffective and the second trial results in HT, or the first two trials are ineffective and the third trial results in HT, and so on.

Therefore, the chance of Inzamam winning is given by

$$\theta(1-\theta) + \eta\theta(1-\theta) + \eta^2\theta(1-\theta) + \eta^3\theta(1-\theta) + \cdots$$
$$= \theta(1-\theta)\frac{1}{1-\eta} = \frac{1}{2} \text{ as } \eta = 1 - 2\theta(1-\theta).$$

Similarly, the other captain Rahul would also win with probability, $\frac{1}{2}$. Thus this new method yields a 'fair toss' irrespective of the coin we use i.e., it does not depend on whether we start with θ different from $\frac{1}{2}$ or even unknown.

It may, however, be noted that our method would not have worked for the Jai and Veeru coin. θ for their coin is 1 and for our method to work we need $0 < \theta < 1$ so that $\theta(1-\theta)$ is greater than 0.

Our method would not have worked for the Jai and Veeru coin. θ for their coin is 1 and for our method to work we need $0<\theta<1$ so that $\theta(1-\theta)$ is greater than 0.

In the existing tossing procedure, we need to toss the given coin only once to decide the winner. What happens in the modified procedure? Although it is not an unending procedure, how many times do we need to toss the given coin to decide the winner? The reader may have many questions at this stage.

Let us now focus on the waiting time distribution. If Y is the number of effective trials required (also called waiting time) to decide the winner, then Y is a geometric random variable. Its distribution (also called waiting time distribution) is specified by the following *probability mass function*:

$$P\left[Y = y\right] = (\eta)^{y-1}\left(1-\eta\right); \; y = 1, 2, 3, \cdots . \qquad (1)$$

Now that we know the distribution of Y, we can address questions of the kind: What is the probability that we need at least, say, M trials to decide the winner? This probability is given by

$$\sum_{y \geq M} P\left[Y = y\right] = \sum_{y \geq M} (\eta)^{y-1}\left(1-\eta\right) = (\eta)^{M-1}.$$

Clearly this probability decreases with M. In fact $\lim_{M\to\infty} (\eta)^{M-1} = 0$.

For example if $\theta = \frac{2}{5}$, i.e., $\eta = \frac{13}{25}$ and $M = 100$, then $(\eta)^{M-1} = \left(\frac{13}{25}\right)^{99} = 7.6618 \times 10^{-29}$, if $M = 10$ then $\left(\frac{13}{25}\right)^{9} = 2.7799 \times 10^{-3}$, if $M = 5$ then $\left(\frac{13}{25}\right)^{5} = 0.03802$. Thus, for example, the probability that we would not

For the 'fair' case, θ =1/2 and η = 1/2, meaning on an average, we would need to perform 2 trials or toss the coin 4 times to decide the winner when we have a fair coin.

need more than 4 trials to decide the winner is at least as high as 0.96.

For the geometric distribution given in (1), $E(Y) = \frac{1}{1-\eta}$. Simply put, on an average we need $\frac{1}{1-\eta}$ trials or equivalently $\frac{2}{1-\eta}$ tosses to decide the winner. For the 'fair' case, $\theta = \frac{1}{2}$ and $\eta = \frac{1}{2}$, meaning on an average, we would need to perform 2 trials or toss the coin 4 times to decide the winner when we have a fair coin.

We have thus far seen how to use a given coin with $\theta \in (0,1)$ to conduct a 'fair toss'. Before we conclude, it may be mentioned in passing that such a coin can also be used to select an individual from among the group of n individuals with equal probability. More generally, it can also be used to select different individuals with arbitrarily specified probabilities. So next time you suspect that the coin being used is not fair then you may resort to the modified toss. Even if your suspicion is unfounded, never mind, the modified toss works as effectively for the fair coin. Is Inzamam listening?

While in Veeru's context it is a clear case of a biased coin, we cannot quite say anything in Inzamam's case. However, it is possible to 'test' whether for any given coin $\theta = \frac{1}{2}$ or not. This falls in the realm of 'testing of hypotheses'. We shall not dwell on this topic here. Our aim here is to devise an algorithm to conduct a fair toss and in no way are we implying that Inzamam was at the receiving end of some evil designs. On the contrary, we do a few computations to find the probability of events like winning not more than 45 tosses out of say 100. Suppose Inzamam captained in 100 matches. Suppose every single toss was conducted using a fair coin (not necessarily the same coin). What is the chance of losing at least 55 tosses? If we assume that tosses are conducted independently using only fair coins, then we have 100 *independent Bernoulli*$(\frac{1}{2})$ random variables. If X denotes the the total number of tosses won by Inzamam, then X is a $Binomial(100, \frac{1}{2})$ random variable. Its *probability mass function* is given by

$$P\left[X = x\right] = \binom{100}{x}\left(\frac{1}{2}\right)^{100};0 \le x \le 100. \qquad (2)$$

We can thus compute the probability $P\left[X \le 45\right]$.

$$P\left[X \le 45\right] = \sum_{x \le 45} P\left[X = x\right] = \left(\frac{1}{2}\right)^{100} \sum_{x \le 45} \binom{100}{x}.$$

This can be shown to be approximately 0.15. This is quite different from 0.

In fact, the probability of winning not more than half the number of tosses, here 50 is $\frac{1}{2}$, which again is a significant number.

Let us make a quick comment on the run of losing tosses. In a 5-match test series consider the string $LLLLL$ which corresponds to Inzamam losing the toss in all five matches and the string $LWLWL$ which corresponds to Inzamam losing the toss in the first third and fifth match and winning the toss in the second and the fourth match. *A priori*, the second string $LWLWL$ may seem to be more probable than the first string $LLLLL$. However, under the assumption of 5 independent fair tosses both strings have exactly same probability, namely $\left(\frac{1}{2}\right)^5$. As a matter of fact all $2^5 = 32$ strings are equiprobable, each having probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32} = 0.03125$.

Appendix.

Here we give a few examples to illustrate how we can generalize the algorithm developed above.

Example 1: Suppose Sumedh's mother has planned a party but she is rather undecided about the dessert. If she is equally keen on offering either mysorep$\overline{a}$k, rôshogolla or gul$\overline{a}$bj$\overline{a}$moon (but only one of them), you can help her out by devising an algorithm using a coin.

Suppose the coin at our disposal, when tossed turns up heads with probability θ, $\theta \in (0,1)$. Now suppose that a trial consists of three (independent) tosses of the given coin then there are $2^3 = 8$ outcomes with different probabilities as shown in the following table:

Outcome	HHH	HHT	HTH	THH
Probability	θ^3	$\theta^2(1-\theta)$	$\theta^2(1-\theta)$	$\theta^2(1-\theta)$

Outcome	HTT	THT	TTH	TTT
Probability	$\theta(1-\theta)^2$	$\theta(1-\theta)^2$	$\theta(1-\theta)^2$	$(1-\theta)^3$

Let us say, Sumedh's mother prepares mysorep$\overline{a}$k if the outcome is HHT, rôshogolla if the outcome is HTH and gul$\overline{a}$bj$\overline{a}$moon if the outcome is THH. If the outcome is any one of HHH, HTT, THT, TTH or TTT then the trial is rendered ineffective and another trial is performed. This process is continued until such times as one of the sweets is chosen.

Let us quickly compute the probability η of any trial being ineffective. Note that the probability $(1-\eta)$ of any trial being effective is $3\theta^2(1-\theta)$.

Therefore $\eta = 1 - 3\theta^2(1-\theta)$.

Again the probability that the procedure terminates is given by

$$(1-\eta) + \eta(1-\eta) + \eta^2(1-\eta) + \eta^3(1-\eta) + \cdots$$

$$= (1-\eta)\left(1 + \eta + \eta^2 + \eta^3 + \cdots\right) = (1-\eta)\frac{1}{(1-\eta)} = 1.$$

Note that Sumedh's mother prepares mysorep$\overline{a}$k if the first trial results in HHT or, the first trial is ineffective and the second trial results in HHT or, the first two trials are ineffective and the third trial results in HHT and so on.

Therefore the chance of mysorepāk being chosen for the dessert is given by

$$\theta^2(1-\theta) + \eta\theta^2(1-\theta) + \eta^2\theta^2(1-\theta) + \eta^3\theta^2(1-\theta) + \cdots$$

$$= \theta^2(1-\theta)\frac{1}{1-\eta} = \frac{1}{3}$$

as $\eta = 1 - 3\theta^2(1-\theta)$. Analogously, the chance of rôshogolla being chosen for the dessert is $\frac{1}{3}$, ditto for gulābjāmoon.

Thus we now have a method of choosing one out of three with equal probability.

Note that since we have to choose one out of three with equal chance we identify a set of at least three outcomes, each having same chance of occurrence and call it an effective set. In our case, the effective set is given by $E_1 = \{HHT, HTH, THH\}$, each outcome having probability $\theta^2(1-\theta)$. There can be more than one such sets. For instance, we could have chosen $E_2 = \{HTT, THT, TTH\}$ as the set of effective outcomes. For the set E_2, each of the outcomes has probability $\theta(1-\theta)^2$. A moment's reflection (?) would tell us that for $\theta > \frac{1}{2}$ it is more efficient to prefer E_1 to E_2, whereas for $\theta < \frac{1}{2}$ it is more efficient to prefer E_2 to E_1. For the set E_1, the probability of a trial being ineffective is $1 - 3\theta^2(1-\theta)$ whereas for the set E_2, the probability of a trial being ineffective is $1 - 3\theta(1-\theta)^2$.

Again a moment's reflection (?) would tell us that if θ were known to be $\frac{1}{2}$ then it would have sufficed to think of a trial with 2 independent tosses.

Note that this algorithm easily generalizes to choosing 1 out of k with equal probability. It would suffice to consider a trial that consists of k independent tosses of the given coin. This can be further improved by choosing smallest integer m such that $\binom{m}{[\frac{m}{2}]} \geq k$, where $[x]$ denotes the integer part of x. It would suffice to consider a trial that consists of m independent tosses of the given coin.

Next we deal with varying probability selection.

Example 2. Continuing with example 1, suppose the order of preference of Sumedh's mother reads 1) gul$\overline{a}$bj$\overline{a}$moon 2) rôshogolla and 3) mysorep$\overline{a}$k. For example, she may want to choose them with different probabilities, say $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ respectively. How can we modify our algorithm?

Note that $\binom{4}{2} = 6$. Now suppose that a trial consists of four (independent) tosses of the given coin then there are $2^4 = 16$ outcomes with different probabilities. A typical outcome is a sequence of length four of letters H and T. There are 6 outcomes in which there are exactly two heads and two tails. Our effective set consists precisely of these 6 outcomes. Let

$$E = \{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}.$$

Each of these 6 outcomes occurs with probability $\theta^2 (1-\theta)^2$.

Let us say, Sumedh's mother prepares gul$\overline{a}$bj$\overline{a}$moon if the outcome is either $HHTT$, $HTHT$ or $HTTH$. She makes rôshogolla if the outcome is either $THHT$ or $THTH$ and she opts for mysorep$\overline{a}$k if the outcome is $TTHH$. If the outcome is not in E then the trial is rendered ineffective and another trial is performed. This process is continued until such times as one of the sweets is chosen.

For this example

$$1 - \eta = 6\theta^2 (1-\theta)^2 \text{ or } \eta = 1 - 6\theta^2 (1-\theta)^2.$$

Clearly the procedure terminates with probability 1.

Note that Sumedh's mother prepares rôshogolla if the first trial results in either $THHT$ or $THTH$ or the first trial is ineffective, and the second trial results in $THHT$ or $THTH$ or the first two trials are ineffective, and the third trial results in $THHT$ or $THTH$ and so on.

Therefore the chance of rôshogolla being chosen for the dessert is given by

$$2\theta^2(1-\theta)^2 + \eta \times 2\theta^2(1-\theta)^2 + \eta^2 \times 2\theta^2(1-\theta)^2 + \eta^3 \times 2\theta^2(1-\theta)^2 + \cdots = 2\theta^2(1-\theta)^2\frac{1}{1-\eta} = \frac{1}{3}$$

as $\eta = 1 - 6\theta^2(1-\theta)$.

Similarly, the chance of mysorep$\overline{a}$k being chosen for the dessert can be shown to be $\frac{1}{6}$ and that of gul$\overline{a}$bj$\overline{a}$moon to be $\frac{1}{2}$.

A moment's reflection would tell us that if θ were known to be $\frac{1}{2}$ then it would have sufficed to consider a trial that consists of three independent tosses of the given fair coin as we get all, $2^3 = 8$, equiprobable outcomes.

In all these situations, as you would have noticed, the basic step is to get hold of an effective set that consists of appropriate number of equiprobable outcomes. This observation would help us to generalize our algorithm to selecting 1 out of k with varying probabilities.

We can also compute, as done in the main section, the expected number of trials required to implement each of these algorithms using the appropriate geometric distribution. It is easy to see that smaller the η (the probability of a trial being ineffective), fewer the trials required, on an average, to implement the algorithm.

Suggested Reading

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