Snippets of Physics

3. Quantum Mechanics on the Run*

T Padmanabhan

T Padmanabhan works at IUCAA, Pune and is interested in all areas of theoretical physics, especially those which have something to do with gravity.

How does one study quantum mechanics in an accelerated frame? The answer to this question leads to some surprising insights into the solutions of the Schrödinger equation for harmonic oscillator!

In any inertial frame ($\bar{S}$), the wave function describing a free particle of unit mass satisfies the standard Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial\bar{x}^2}, \tag{1}$$

where we have used units in which $\hbar = 1$ and have confined our attention to one dimension with a coordinate $\bar{x}$. Consider now another observer (S), who is moving along the x-axis in an arbitrary manner. This observer can use a coordinate x with $\bar{x} \equiv x + l(t)$, where $l(t)$ is an arbitrary function of time. (In the spirit of non-relativistic mechanics, we assume that the time coordinate is the same for both the observers.). Viewed from $\bar{S}$, the origin of S moves with the trajectory $l(t)$. What happens to the Schrödinger equation and its solutions in this accelerated frame? Somewhat surprisingly, standard textbooks do not discuss this issue. It turns out that the coordinate transformations to non-inertial frames provide some interesting insights into quantum mechanics which we will discuss in this installment.

One way to attack this problem is as follows: We know that the free particle Lagrangian $\bar{L} = (1/2)\dot{\bar{x}}^2$ expressed in terms of the x coordinate is just $\bar{L} = (1/2)(\dot{x}+\dot{l})^2$ in S. (The overdot denotes the time derivative.) Expanding the square and manipulating the expression a little, we

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can rewrite the Lagrangian in the form

$$\bar{L} = \frac{1}{2}\dot{x}^2 - \ddot{l}(t)x + \frac{\mathrm{d}}{\mathrm{d}}t\left[K + x\dot{l}\right] \equiv L + \frac{\mathrm{d}f}{\mathrm{d}t}; \quad \dot{K} = \frac{1}{2}\dot{l}^2. \tag{2}$$

In classical mechanics, we know that two Lagrangians related by the $\bar{L} = L + (\mathrm{d}f(x,t)/\mathrm{d}t)$ will lead to the same equations of motion. So, the accelerated observer could ignore the $(\mathrm{d}f/\mathrm{d}t)$ term in the Lagrangian in (2) and just use the Lagrangian $L = (1/2)\dot{x}^2 - \ddot{l}(t)x$ to describe the physics. This makes sense, because L describes a particle moving under the action of a (time dependent) force $a(t) \equiv \ddot{l}(t)$. This is precisely the pseudo-force one expects to see in the accelerated frame.

The corresponding quantum theory, based on the Lagrangian L, however, will lead to the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + a(t)x\psi \tag{3}$$

which, in general, can be difficult to solve. Our first task is to understand the physics behind this equation.

To do this, let us take a closer look at the quantum mechanics based on the two Lagrangians related by a total time derivative, like $\bar{L} = L + (\mathrm{d}f(x,t)/\mathrm{d}t)$. The extra term *does* change the form of canonical momentum $[p = \partial L/\partial\dot{x}]$ and Hamiltonian $[H = p\dot{x} - L]$. Using $(\mathrm{d}f/\mathrm{d}t) = (\partial f/\partial t) + \dot{x}(\partial f/\partial x)$, it is easy to see that the new momentum and Hamiltonian are related to the old ones by:

$$\bar{p} = p + (\partial f/\partial x) \equiv p + f'; \quad \bar{H} = H - (\partial f/\partial t) = H - \dot{f}. \tag{4}$$

Transformation of Schrödinger equation to non-inertial frames can provide same interesting insights to quantum mechanics.

What happens to a solution $\psi(t,x)$ of the Schrödinger equation in quantum theory, when we go from L to $\bar{L}$? Clearly, ψ has to change such that the expectation values of the operators $p = -i\partial/\partial x, H = i\partial/\partial t$ change according to equation (4). It is easy to see that this requires ψ to be modified by a phase factor: $\psi \to \bar{\psi} = e^{if}\psi$. For example,

$$\langle p\rangle_{new} = \int \mathrm{d}x\bar{\psi}^* \left[-i(\partial/\partial\bar{x})\right]\bar{\psi}$$

$$= \int \mathrm{d}x\psi^* \left[e^{-if}[-i(\partial/\partial x)]e^{if}\right]\psi = \langle p\rangle_{old} + \frac{\partial f}{\partial x}, \quad (5)$$

where we have used the standard result:

$$e^{-if}[-i(\partial/\partial x)]e^{if} = -i(\partial/\partial x) + f'. \quad (6)$$

[Note that $(\partial/\partial\bar{x})_t = (\partial/\partial x)_t$, etc.] Thus the addition of a total time derivative to the Lagrangian, $L \to \bar{L} = L + (\mathrm{d}f/\mathrm{d}t)$, leads to the wave function picking up a phase factor $\psi \to \bar{\psi} = e^{if}\psi$.

We can use these results to solve the Schrödinger equations of the form in (3) with an arbitrary time-dependent function $a(t)$! Suppose we have solved the Schrödinger equation (1) for a free particle and it has a solution $\bar{\psi}_{\text{free}}(\bar{x}, t)$. The corresponding solution in the accelerated coordinates, of course, can be obtained by just a shift $x \to \bar{x} = x + l(t)$, so that $\bar{\psi} = \bar{\psi}_{\text{free}}(x + l, t)$ in the new coordinates. But since, $\bar{L}$ is related to L by the addition of a total time derivative term, the solution $\bar{\psi}$ differs from the solution ψ of (3) only by a phase factor! That is, $\bar{\psi} = \bar{\psi}_{\text{free}}(x + l, t) = \mathrm{e}^{if}\psi$. Thus we can now write a solution to (3) as a free particle solution with a shift in x and an addition of a phase:

$$\psi(t, x) = \bar{\psi}_{\text{free}}\left[x + l(t), t\right] \exp -i(K + x\dot{l}), \quad (7)$$

where $\bar{\psi}_{\text{free}}$ is any solution to the free particle Schrödinger equation. [Of course, you can directly verify that the function in (7) solves (3).] Hence you can now solve (3) for any given $a(t)$!

You will also notice that $|\psi|^2 = |\psi_{\text{free}}(x + l(t), t)|^2$, so that the probability just gets shifted by the classical trajectory $l(t)$ as we would have expected. If ψ_{free} represents, say, a dispersing wave packet centered around the origin, the new probability distribution will represent a wave packet moving along the trajectory $-l(t)$ with the same dispersion.

We will now apply this result to two simple examples, to illustrate its power. As a first example, consider a particle moving in a uniform force field with a = constant. The Hamiltonian $H = (1/2)p^2 + ax$ is time independent and hence allows for stationary states which satisfy $H\phi_E = E\phi_E$. The eigenfunction ϕ_E, however, happens to be Airy function[1] in x-space. This, however, is one problem in which the momentum space representation of the operators with $x = i(\partial/\partial p)$ turns out to be easier to handle! The Schrödinger equation in the p-representation is now $ia(\partial\phi/\partial p) = (E - p^2/2)\phi$. Integrating this equation and then Fourier transforming we get the solution in the x-representation to be

$$\phi_E(x) = \int_{-\infty}^{\infty} \mathrm{d}p \exp i[p(x - E/a) + (p^3/6a)], \qquad (8)$$

which is indeed an integral representation for the Airy function [1].

Let us now solve the same problem by our approach. We begin with the simplest free particle solution to the Schrödinger equation, which are the momentum eigenfunctions $\psi_{\text{free}}(t, x) = \exp(-ipx + ip^2t/2)$. We next obtain the solution to (3) by the simple transformation $x \to x + l(t)$, where $\ddot{l} = a$ = constant and the addition of a phase as indicated in (7). This gives the solution:

$$\psi = \exp -i[x(p - at) + (1/2)pat^2 - (1/2)p^2t - (1/6)a^2t^3]. \qquad (9)$$

(You can directly verify that this function satisfies (3)). This is, of course, not an energy eigenfunction. However, a Fourier transform of this expression with respect to t

$$\phi_E(x) = \int_{-\infty}^{\infty} \mathrm{d}t \ \psi(t, x) \exp iEt \qquad (10)$$

[1] Airy function is a special function Ai(x) which is a solution of the equation $y'' - xy = 0$.

This analysis provides a way of understanding the coherent excited states of a harmonic oscillator.

will give the energy eigenfunctions for a particle moving in a uniform force field. Changing the variable of integration from t to $\xi \equiv (at - p)$, you will find that various terms cancel out nicely, leading to

$$\phi_E(x) \propto \int_{-\infty}^{\infty} \mathrm{d}\xi \ \exp i[\xi(x - E/a) + (\xi^3/6a)], \qquad (11)$$

which are the same energy eigenfunctions as in (8) except for an unimportant phase.

Our approach also leads to another interesting result, which occurs in the case of the quantum harmonic oscillator. The ground state of a harmonic oscillator is described by a Gaussian wave function with the probability distribution $|\phi_0(x)|^2 \propto \exp[-\omega x^2]$. We also know that the harmonic oscillator admits coherent states with the probability distribution $|\phi_A(x)|^2 \propto \exp[-\omega(x - A\cos\omega t)^2]$, which is obtained by just shifting the ground state probability distribution by $x \to x - A\cos\omega t$. What is more suprising is that such coherent states exist *even for the excited states* of the oscillator with the same shift! The existence of such states is a bit of a mystery in the conventional approach to quantum mechanics but our analysis gives an interesting insight into this issue.

To understand this, let us apply the transformation $x \to \bar{x} = x + l(t)$ to the harmonic oscillator Lagrangian $L = (1/2)(\dot{x}^2 - \omega^2 x^2)$. Elementary algebra shows that the new Lagrangian has the structure

$$\bar{L} = (1/2)(\dot{x}^2 - \omega^2 x^2) - (\ddot{l} + \omega^2 l)x + \frac{\mathrm{d}f}{\mathrm{d}t}, \qquad (12)$$

where f is again a function determined by $l(t)$ but its explicit form is not important. Let us now choose $l(t)$ to be a solution to the classical equation of motion $\ddot{l} + \omega^2 l = 0$. To be specific, we will take $l = -A\cos\omega t$. If you want, you can think of this as shifting to a frame which is oscillating with the particle. We then see that the second term in (7) vanishes and $\bar{L}$ has the *same* form as the original harmonic oscillator Lagrangian except for the total derivative. (This miracle occurs only for the quadratic potential!) The solutions to the Schrödinger equation are, therefore, the *same* as the standard solutions to the harmonic oscillator problem with a shift $x \to x + l(t)$ and an extra phase factor!

Furthermore, the probabilities do not care for the phase factor and we have the result $|\bar{\psi}|^2 = |\psi(x + l(t), t)|^2$. If ψ is the ground state then this shift leads to the standard coherent state. But if you take the nth excited state of the oscillator $\psi_n(x, t)$, shift the coordinate and add a phase, then we get another valid solution $e^{if}\psi_n(x - A\cos\omega t, t)$. As far as the probability goes, $|\psi_n(x - A\cos\omega t, t)|^2$ merely traces the original probability distribution with the mean value oscillating along the classical solution. In our approach, we see that a harmonic oscillator gets mapped back to a harmonic oscillator when we move to a frame with $\ddot{l} + \omega^2 l = 0$ with just a shift in x (and a phase which is irrelevant for the probabilities). That is why such coherent states exist even for the *excited* states of the harmonic oscillator. Hopefully our analysis makes this result somewhat more transparent.

Address for Correspondence
T Padmanabhan
IUCAA, Post Bag 4
Pune University
Campus
Ganeshkhind
Pune 411 007, India.
Email:
paddy@iucaa.ernet.in
nabhan@iucaa.ernet.in

Suggested Reading

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