The dynamical laws for physical systems are usually expressed in the form of differential equations. They describe the evolution of a given system over infinitesimal intervals of time. However from the early days of mechanics an alternative view of motion, which looks directly at evolution over finite intervals of time, has been studied by many. Such an approach in optics is also well known. In the essay reproduced here Planck gives a masterful survey of the Principles of Least Action in mechanics and evaluates the contributions of the pioneers from Maupertuis through Lagrange to Hamilton in a critical fashion. For interested students and teachers the following references are valuable: Cornelius Lanczos, *The Variational Principles of Mechanics*, University of Toronto Press (1966); the *Feynman Lectures on Physics*, Vol.II, Chapter 19. The latter is available in an Indian edition.

N Mukunda

The Principle of Least Action

Max Planck

As long as physical science exists, the highest goal to which it aspires is the solution of the problem of embracing all natural phenomena, observed and still to be observed, in one simple principle which will allow all past and, especially, future occurrences to be calculated. It follows from the nature of things, that this object neither has been, nor ever will be, completely attained. It is, however, possible to approach it nearer and nearer, and the history of theoretical physics shows that already an extensive series of important results can be obtained, which indicates clearly that the ideal problem is not purely Utopian, but that it is eminently practicable. Therefore, from a practical point of view, the ultimate object of research must be borne in mind.

Among the more or less general laws, the discovery of which characterize the development of physical science during the last century, the principle of Least Action is at present certainly one which, by its form and comprehensiveness, may be said to have approached most closely to the ideal aim of theoretical inquiry. Its significance, properly understood, extends, not only to mechanical processes, but also to thermal and electro-dynamic problems. In all the branches of science to which it applies, it gives, not only an explanation

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of certain characteristics of phenomena at present encountered, but furnishes rules whereby their variations with time and space can be completely determined. It provides the answers to all questions relating to them, provided only that the necessary constants are known and the underlying external conditions appropriately chosen.

This central position attained by the principle of least action is, however, not even to-day quite undisputed; for a long time the principle of conservation of energy has been a keen competitor. The latter governs in a similar manner the entire range of physics and certainly possesses the advantage of being more easily explained. It is, therefore, advisable to examine briefly the relative positions of these two principles.

The principle of conservation of energy can be derived from the principle of least action and is consequently contained in it. The converse is, however, not true. Accordingly, the former is the more particular, and the latter the more general principle. As an illustrative example, let us consider the motion of a free particle under no forces. According to the principle of conservation of energy, such a particle moves with constant velocity, but nothing is said concerning the direction of the velocity, since kinetic energy does not depend on direction. The path of the particle could, for example, be rectilinear or curvilinear. On the other hand, the principle of least action demands, as we shall show in detail below, that the particle must move in a straight line.

Now in this simple example an attempt could be made to extend the principle of conservation of energy by making certain simple assumptions, such as that not only the total kinetic energy of the moving particle remain constant, but that also the component of the energy along a certain given direction in space be constant. Such an extension would be foreign to the principle of energy and would be difficult to apply to more general problems. In the case of the spherical pendulum, that is a heavy particle in motion on a fixed sphere, this principle could only furnish the following solution. During the upward motion, the kinetic energy decreases in a certain manner, and increases during the downward motion. The path of the particle cannot however be determined, whereas the principle of least action completely solves all questions bearing on the motion.

The reason for the difference in the results derived from the two principles lies in the fact that when applied to any problem, the principle of conservation of energy furnishes one equation only, while it is necessary to obtain as many equations as there are independent variables in order to determine the motion completely. Thus in the case of the free particle three equations are needed, and in that of the spherical pendulum two. Now, the principle of least action furnishes, in every case, as many equations as there are variables. Moreover, it enables several equations to be embraced in one formula, as it is a variations principle, as opposed to the principle of energy.

From the infinite number of virtual motions imaginable under the given conditions, it indicates a quite definite motion by means of a simple criterion, and shows that this is the actual motion. The criterion is that in the transition from the actual motion to an arbitrary motion infinitesimally close to it, or more accurately, for every infinitesimally small variation of the given motion consistent with the given conditions, a certain function characteristic of the variation vanishes. By this means one equation is derived from every independent variable, as in the case of a maximum or minimum problem.

Now, it must be understood that the principle of least action only attains a definite significance when we have given to us the prescribed conditions with which the virtual motions must be consistent, as well as the characteristic functions which vanish for every arbitrary variation of the actual motion. The problem of determining the correct conditions always forms the essential difficulty in formulating the principle of least action. However, it must be obvious that the idea of combining into one variations principle a number of equations, necessary for defining the motion of any complicated mechanical system is in itself of great importance and represents an appreciable advance in theoretical research.

In this connection mention may certainly be made of Leibniz's theorem, which sets forth fundamentally that of all the worlds that may be created, the actual world is that which contains, besides the unavoidable evil, the maximum good. This theorem is none other than a variations principle, and is, indeed, of the same form as the later principle of least action. The unavoidable combination of good and evil corresponds to the given conditions, and it is clear that all the characteristics of the actual world may be derived from the theorem, even to the details, provided that, on the one hand the standard for the quantity of good, and on the other hand the given conditions, be rigidly defined along mathematical lines – the second is just as important as the first. Before, however, we can hope to derive important results from the principle we must advance still further. First of all, the characteristic quantity, which vanishes in the case of the actual motion, must be investigated and understood. We may proceed from two different points of view. According to one, the characteristic function is referred to an isolated time point or to an infinitesimal time element; according to the other, it is referred to a finite time interval during the motion. We arrive at two different classes of variations principles according as to whether we decide to adopt the one or the other standpoint.

To the first class belong Bernoulli's Principle of Virtual Displacements, d'Alembert''s

Principle of Resistance, Gauss's Principle of Least Constraint, and Hertz's Principle of the Shortest Path. All these principles may be considered as differential principles in so far as they apply the characteristic criterion to a property of the motion which is referred to an isolated movement or to a small element of time. In the case of mechanical systems any one of them is completely equivalent to any other and to Newton's Laws of Motion. But they all suffer the disadvantage that it is only for mechanical systems that they have any significance, and their exposition renders it necessary to introduce special point co-ordinates for the mass systems under consideration, and varies with the choice of co-ordinates. The exposition is usually comparatively complicated.

The inconvenience of such mechanical systems of co-ordinates may be overcome if, as a matter of course, the variations principle be considered as an Integration Principle by referring it to a finite time interval. Then, of all virtual motions, the actual motion is that defined by the property that for any permissible variation from it a certain time integral vanishes. In the most important cases, this condition can be expressed as follows: For the actual motion, a certain time integral, which may be called "The Magnitude of the Action"* or the "Action"+; of the motion, is less than that of any other motion consistent with the prescribed conditions. Thus, according to Leibniz, the action of any single material particle is equal to the time integral of the kinetic energy, or, in other words, the time integral of the velocity.

In this manner, the principle of least action can be applied without reference to any special system of co-ordinates, and without pre-supposing any mechanical phenomenon, since only time and energy appear in the expressions. A special feature appears through the introduction of the time integral, and the presence of this feature has always been, and is even now, considered by many physicists and philosophers to be a criticism to be levelled against the principle of least action as against every other variations principle. Thus, by referring to a finite time interval, the motion at any instant is investigated with the help of a later motion, and present events are in a certain manner made dependent upon later events, and the principle acquires a teleological character. When dealing with the principle of causality, it must be possible to understand and derive all the characteristics of a motion from previous circumstances, without reference to anything that may happen later. This appears not only feasible but a direct logical consequence. On the other hand, when seeking the most lucid relations in the system of natural laws, such aids as reference to later events will be considered permissible in the interests of the desired harmony. These may not be directly

* Wirkungsgrösse +Aktion

essential for the complete exposition of natural phenomena, but they might, perhaps, be more convenient to handle, or more easily interpreted. I would remind you that in order to retain the symmetry of equations in mathematical physics, it often happens that the quantities to be determined are not expressed in terms of the independent variables themselves. On the contrary, one or more superfluous variables are often introduced in order to utilize the great practical advantages of symmetry.

Since Galileo's time, physics has achieved its greatest successes by rejecting all teleological methods. It is justified, therefore, in definitely opposing all attempts at introducing teleological points of view into the law of causality. However, though the introduction of a finite time integral is unnecessary for formulating the laws of mechanics, yet the integral principle should not, as a matter of course, be rejected. The question of its correctness has nothing to do with teleology. It is far more a practical question, and may be thus expressed: does the exposition of the laws of Nature by means of the integral principle accomplish more for the purposes of theoretical physics than other expositions? From the modern standpoint, the answer must be in the affirmative, on account of the fact that the integral principle, as already mentioned, is independent of any special co-ordinates. The modern principle of relativity provides, as we shall see later, a complete explanation, not only of the practical significance, but also of the need for introducing the finite time interval into the fundamental principles of mechanics.

In the exposition already given of the principle of least action, no account has been taken of the prescribed conditions of the virtual motions. These are, however, quite as important as the magnitude of the *action* itself, for the significance of the principle differs with the nature of the prescribed conditions. It is not only a question of how the selection is made, but also of the nature of the motions determined by the choice. This circumstance was at first overlooked, and many serious errors were thereby introduced. It was a long time before it was clearly explained, and the principle of least action correctly understood. If the principle be said to have been discovered at this time, the honour should be given to Lagrange. This, however, would be an injustice to other men who had prepared the way for Lagrange to bring the work later to a satisfactory completion. Of these, the first was Leibniz; indeed, he was the chief, according to a letter dated 1707, the original of which has been lost. Then came Maupertuis and Euler. It was chiefly Moreau de Maupertuis (appointed president of the Prussian Academy of Sciences (1746-1759) by Frederick the Great) who not only recognized the existence and significance of the principle, but used his influence in the scientific world and elsewhere to procure its acceptance. Maupertuis repeatedly announced in different forms, his principle of *Mitwelt*, and zealously defended it against what were often authoritative criticisms. The zeal with which he did this rose at times to fanaticism, and was quite disproportionate to the scientific value of the enunciation considered most suitable by him. It is impossible to reject the idea that his energetic adherence to really unattainable theses, arose not alone from scientific conviction, but at least equally from a firm intention to ensure for himself, at all costs, the prior claim to the discovery of what he regarded as his most important work. This is especially shown in the passionate attempts he made to dispute Leibniz's letter (already referred to) when it was produced by Professor Samuel Kunig in 1751 – attempts which almost led him to abuse the high position he occupied. Human weakness and vanity have hardly ever been more severely punished than in the case of the president of the Berlin Academy. His varying fortunes, which occasionally induced the great royal philosopher to interfere, have been repeatedly described in detail by historians and in technical literature by A. Mayer (1877), H. von Helmholtz (1887), E. du Bois-Reymond (1892), and H. Diels (1898). An account of the discussion, from the standpoint of the general development of mathematical science, is given in Cantor's "History of Mathematics," and its significance for the Berlin Academy is dealt with in Harnack's history of the Academy.

Maupertuis's exposition of the principle of least action asserted no more than "that the action applied to bring about all the changes occurring in Nature is always a minimum." Strictly, this formulation does not admit any conclusions to be drawn regarding the laws governing the changes, for as long as no statement of the conditions to be satisfied is made, no deductions can be made as to how the variations are balanced. Maupertuis had not the faculty of analytical criticism necessary to discern this want. The failure will be more easily understood when it is realized that Euler himself, a brilliant mathematician, did not succeed in producing a correct formulation of the principle, though he was assisted by many colleagues and friends.

Maupertuis's real service consisted in his search for a principle that would be, above all, a minimum principle. That was the real object of his investigation. To this end he made use of Fermat's principle of quickest arrival, although its bearing upon the principle of least action was very indirect and, at all events, unknown to the physics of his time. The interest in the principle of least action was fundamentally based upon the metaphysical idea that the rule of the Deity reveals itself in Nature. Therefore, every natural occurrence is founded on an intention which is directed to a certain end, and which indicates the most direct way and the most suitable methods towards attaining this end.

How inadequate, and even misleading, teleological methods can be, is best realized from the fact that, from a general point of view, the principle of least action is not, strictly speaking, a minimum principle at all. Thus, the statement that the path of a particle, free to move without friction on a sphere, is the shortest line joining its initial and final positions, is not true if the path is longer than the semi-circumference of a great circle on the sphere. Beyond the semi-circumference, therefore, Divine foresight cannot operate. Still more striking is the fact that when considering non-holonomous systems, the virtual motions bear no relation to the possible motions, and thus the minimum condition loses all its significance.

In spite of all this, however, it must be borne in mind that the strong conviction of the existence of a close relation between natural laws and a higher will has provided the basis for the discovery of the principle of least action. Provided, of course, that such a belief is not confined within too narrow limits, it certainly does not admit of proof, but, on the other hand, it can never be disproved, for then one could ultimately ascribe any contradiction to an incomplete formulation.

J. L. Lagrange was the first to express correctly the principle of least action (1760). Thus – of all the motions that may bring a system of material particles from a certain initial position to a given final position (the total energy remaining constant), the actual motion is that for which the action is a minimum. The virtual motions must, therefore, satisfy the principle of energy. They may, on the other hand, take any arbitrary time. According to this conception, the path of a particle is that along which it will reach its final position in the shortest time, if it move with constant velocity, and if frictional forces be absent. Thus, the path is the line of shortest length, that is, for a free particle a straight line.

Later, C. G. J. Jacobi and W. R. Hamilton showed that the principle admitted of other representations. Hamilton's exposition was of great importance from the standpoint of future developments. According to him, the total energy of the virtual motions to be compared need not remain constant, but the motions must take place in the same time. Then, however, the action which is a minimum for the actual motion, must not be expressed by Maupertuis's time integral of the kinetic energy, but by the time integral of the difference of kinetic and potential energies. Applying this method to the above example of a particle in motion when not affected by frictional forces, the principle shows that of all the possible curves, the actual path is that along which the particle reaches its final position in a given time with the least velocity, again the line of shortest length.

In a characteristic way, the principle of least action did not at first exercise an appreciable effect on the advance of science, even after Lagrange had completely established it as a part of mechanics. It was considered more as an interesting mathematical curiosity and an unnecessary corollary to Newton's laws of motion. Even in 1837 Poisson could only call it "a useless rule". It was in the investigations of Thomson and Tait, G. Kirchhoff, C. Neumann, L. Boltzmann and others that the principle first proved itself to be an excellent method for solving problems in hydrodynamics and elasticity. While the usual methods of mechanics sometimes worked with difficulty and at other times refused to work, a revolution was in the making – the value of the principle began to be realized. In 1867 Thomson and Tait wrote concerning it, "Maupertuis's celebrated principle of least action has been, even up to the present time, regarded rather as a curious and somewhat perplexing property of motion than as a useful guide in kinetic investigations. We are strongly impressed with the conviction that a much more profound significance will be attached to it, not only in abstract dynamics, but in the theory of the several branches of physical science now beginning to receive dynamic explanation."

It was also shown that when applying the principle, especially when defining the prescribed conditions of the virtual motions, particular care was necessary if errors were to be avoided. For example, when considering the irrotational motion of an inviscid fluid round a solid body, it is, in general, not sufficient to assume the initial and final positions of the body given, the initial and final positions of the fluid elements must also be given. H Hertz made an error of another type, in the introduction to his mechanics, when he applied the principle of least action to investigate the motion of a sphere rolling on a horizontal plane, and assumed for the virtual displacements certain conditions not allowed when dealing with non-holonomous systems. O. Hölder and A. Voss did much to make this problem clear.

The fundamental significance of the principle of least action, as a general principle, was first understood when it was realized that it could be applied to systems whose mechanism was either entirely unknown, or so complicated that they could not be considered by means of ordinary systems of co-ordinates. After L. Boltzmann and later R. Clausius had perceived the close relation between the principle and the second law of thermo-dynamics, H. von Helmholtz gave, for the first time, a complete and systematic summary of such applications of the principle as were possible at the time to the three great branches of physics – mechanics, electro-dynamics, and thermo-dynamics.

This was a surprising achievement in view of the comprehensiveness of the range covered. For his calculations, Helmholtz chose Hamilton's form of the principle as being the most convenient, and made some extensions of a formal nature. He used the term kinetic potential to denote the quantity the time integral of which was what Hamilton called the action. He thus retained the hypothesis that the principle was fundamentally a mechanical one. This limitation, however, was somewhat of a retrogression, since it was not necessary to consider the mechanical constitution of several of his systems, such as galvanic streams and magnets. On the other hand, Helmholtz accomplished the deciding act, in that he did not derive his kinetic potential from the difference of the kinetic and potential energies as had hitherto been done, but he set forth the kinetic potential as the primary quantity, and thus determined the magnitude of the energy and all the remaining laws of motion.

The chief consequence of this new method of considering the question was an immediate generalization of some importance. The kinetic potential is not only the analytical form of the energy, but gives also its magnitude, which varies according to the choice of the independent variable. For example, some of the equations of motion can be used in order to reduce the number of independent variables. The variables eliminated have then disappeared from the principle entirely and may be said to correspond to the concealed motion. In each such case, the kinetic potential assumes a different magnitude, and thus are explained, for example, the expressions derived for the potential in thermo-dynamics which differ with the choice of the independent variable. Helmholtz showed how these different expressions are interrelated, and follow from one another; he also showed that the kinetic potential can assume a form in which it appears no longer as the difference between the kinetic and potential energies. This result demonstrated at once the universality of the principle, for outside the range of mechanics the distinction between kinetic and potential energies is no longer possible, and, therefore, the possibility of deriving the kinetic potential uniquely from the energy disappears, while in each case the converse is simple.

Although it had been possible for Helmholtz to hold fast to the assumption (at least in principle) that all physical phenomena can ultimately be reduced to the motions of simple particles, considerable doubt has since been thrown on the validity of the assumption, at least as far as electro-dynamics is concerned. There is no doubt, however, from all the results hitherto obtained, that the principle of least action has been proved to be applicable and useful in physics outside the range of mechanics, especially in the electro-dynamics of absolute vacuum. Without making use of any mechanical hypothesis, J. Larmor (1900), H. Schwarzschild (1903), and others have derived the fundamental equations of electro-dynamics and electron theory from the Hamiltonian principle.

Thus the development of the principle of least action has followed along similar lines to that of the principle of conservation of energy. The latter was also originally regarded as a mechanical principle, indeed its general validity was directly considered, for a long time, to be a suitable basis of the mechanical view of Nature. To-day, the mechanical conception of Nature has lost ground, while there has never been any occasion to doubt the universality of the principle of energy. Anyone desiring to regard the principle of least action as mechanical would to-day have to apologize for doing so.

The most brilliant achievement of the principle of least action is shown by the fact that Einstein's theory of relativity, which has robbed so many theorems of their universality, has not disproved it, but has shown that it occupies the highest position among physical laws. The reason for this is that Hamilton's "Action" (not Maupertuis's) is an invariant with respect to all Lorentz transformations, that is, it is independent of the system of reference of the observers. This fundamental characteristic gives a far-reaching explanation of the striking circumstance (and at first sight unfortunate) that "action" refers to a time interval, and not to an instant of time. In the theory of relativity time plays a part analogous to space. According to the theory of relativity the problem of determining the state of a system of bodies in different positions at any time from the state in different positions at any given time is exactly similar to the problem of determining the state at different times in all positions from the state of the system at different times in any given position. Though the first problem is usually regarded as the real problem of physics, yet, strictly speaking, there is in it a certain arbitrariness and unreal limitation, which only finds a historical explanation in the fact that its solution is, in the majority of cases, of greater use to humanity than that of the second. Now, just as the calculation of the action of a system of bodies necessitates an integration over the space occupied by the bodies, the action must contain also a time integral in order that no priority is given to space over time, for space and time together constitute the universe to which the action relates.

As in the case of the principle of least action, the principle of the conservation of energy has also a special position in the theory of relativity. Energy is, however, not an invariant with reference to Lorentz transformations any more than it was earlier with respect to Galileo's transformations. For in energy time plays the more important part. The corresponding principle into which space enters is the principle of conservation of momentum. The principle of least action stands superior to both, even when considered together, and it appears to govern all the reversible processes of Nature. Nevertheless, it offers no explanation for irreversibility, since according to it, all phenomena can proceed backwards or forwards in any direction in space and time. That is why the problem of irreversibility has not been considered in this paper.