Is There a Pattern?

Adam Skalski

One. Mathematics is the language of nature. Two. Everything around us can be represented and understood through numbers. Three. If you graph these numbers, patterns emerge.

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The words above constitute a credo of one of the strangest and at the same time most fascinating fictional mathematical characters portrayed by the modern cinema. His name is Max Cohen and he is the lead character in the American film 'π'. In the movie Max is searching for a mysterious numerical pattern which is supposed to be a key to the physical laws ruling all aspects of the universe. The starting point of this article is neither to follow Max's quest – which is rather unreasonable – nor to discuss the first or the second statement of his program – both of them being very tempting, but somewhat difficult to prove. We would like to look at a very specific, physically meaningful number and show that in a sense this number contains 'no patterns'. The number in question is the same number which is the source of fascination for Max. It is π – the ratio of the perimeter of a circle to its diameter or, equivalently, the area of a disk of radius 1 (actually the symbol π originates from the word 'perimeter' and was first used in the 17th century). We all know that

$$\pi \approx 3.1415926535\ldots .$$

Several approximate expressions for π have been known for millennia – the ancient Egyptians and Babylonians knew and used some (admittedly rougher) estimates of the type given above. But can we write an explicit formula? Does the decimal expansion ever end? If not, may be there is a 'pattern' – may be from some point on the decimal expansion is given by some finite sequence of digits repeated infinitely many times? If either of the

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If π were equal 3, this sentence would look like that.

Figure 1.

last two statements was true, π would have to be *rational*, that is it would have to be equal to a fraction $\frac{m}{n}$ where m and n are integers ($n \neq 0$).

It is very easy to see that if the decimal expansion of a number is finite, then it is rational: for example 5.345637 $= \frac{5345637}{1000000}$. If the expansion is infinite but periodic, a simple argument shows that the number still has to be rational. Again, to avoid complicating a straightforward statement, we will consider an example: suppose that $x = 3.2456456456... = 3.2(456)$ (the second expression is just a notation). Note that we have then $x = \frac{32}{10} + \frac{1}{10}0.(456)$ and if $y = 0.(456)$ then $1000y = 456 + y$, so that $y = \frac{456}{999}$. In conclusion, $x = \frac{32}{10} + \frac{456}{9990}$ and so is obviously rational. As the argument above applies in general, we have established the following statement:

Statement 1. *Every number which has a finite or periodic decimal expansion is rational.*

Decimal system clearly does not play any important role here, the same holds true if we consider binary, hexadecimal or any other expansions based on considering powers of a fixed integer greater than 1.

Before we continue our quest of looking for patterns in π we will still spend some time looking at some other general examples. Can one easily exhibit a real number which is not rational (such numbers are known under a slightly offensive name of *irrational*)? Most (if not all) of the readers would know that $\sqrt{2}$ is irrational. This is again easy to prove: suppose that $m, n \in \mathbb{Z}$ (here and in what follows we will use $\mathbb{Z}$ to denote the set of all integers, $\mathbb{N}$ for the set of all positive integers) are such that $\sqrt{2} = \frac{m}{n}$. We can assume that both m, n are positive and that $n \neq 1$ (can you see why? It is almost trivial, but is in fact key to what follows). Then $m^2 = 2n^2$ and therefore m has to be even. But if m is even then $m = 2m_1$ for some $m_1 \in \mathbb{N}$ and we have $n^2 = 2m_1^2$, so that n has to be even, $n = 2n_1$ for some $n_1 \in \mathbb{N}$. But then $\sqrt{2} = \frac{m_1}{n_1}$ and we can repeat the game

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with m_1 and n_1 (with $n_1 \neq 1$ again). After k steps we obtain the existence of $m_k, n_k \in \mathbb{N}$ such that $m = 2^k m_k$, $n = 2^k n_k$. This is contradictory, as it would imply that $m \geq 2^k$ for each $k \in \mathbb{N}$. Reaching contradiction we have shown that $\sqrt{2}$ is irrational. Are all square roots of positive integers irrational? (I refer to the article by V G Tikekar in the December 2007 issue of *Resonance* for six more proofs of this fact). Well, not quite, as for example $\sqrt{4} = 2$. However, again a close analysis of the proof given above allows to deduce the following:

Are all square roots of positive integers irrational?

Statement 2. *If $n \in \mathbb{N}$ then $\sqrt{n}$ is either integer or irrational.*

Is π rational? It seems that already Aristotle guessed it is not, but for the first formal proof of this fact one had to wait till the second half of the 18th century, when a Swiss mathematician J H Lambert established this via a complicated method using continuous fractions. Lambert's proof was soon after simplified and popularised by A M Legendre, but still remained far from elementary. The beautiful proof we will demonstrate below was given in 1947 by I Niven in [1]. It uses only basic properties of derivatives, integrals and limits of sequences of real numbers.

We need first some basic information about the sinus function: $f(x) = \sin(x)$. The function f is continuous, $f(x) > 0$ for all $x \in (0, \pi)$ and moreover $f(0) = f(\pi) = 0$. Consider now the family of polynomials $\mathcal{P}$ defined as follows: A polynomial P (with real-valued coefficients) belongs to $\mathcal{P}$ if and only if $P(0)$, $P(\pi)$, $P'(0)$, $P'(\pi)$, $P^{(2)}(0)$, $P^{(2)}(\pi), \ldots$ are all integers. The class $\mathcal{P}$ has several interesting properties:

(i) If $P(x)$, $Q(x)$ are in $\mathcal{P}$ then $P(x)Q(x)$ also belongs to $\mathcal{P}$. This is easy to see from the formula for the k-th derivative of the product:

$$(PQ)^{(k)}(x) = \sum_{l=0}^{k} \binom{k}{l} P^{(l)}(x) Q^{(k-l)}(x).$$

Is π rational?

(ii) If $P(x)$ is in $\mathcal{P}$ then the integral $\int_0^\pi f(x)P(x)\mathrm{d}x$ is an integer. This is a consequence of the integration by parts formula, if the degree of $P(x)$ is say n then we have

$$\int_0^\pi \sin(x)P(x)\mathrm{d}x = \int_0^\pi \cos(x)P'(x)\mathrm{d}x + [-\cos(x)P(x)]_0^\pi$$
$$= -\int_0^\pi \sin(x)P''(x)\mathrm{d}x + [\sin(x)P'(x) - \cos(x)P(x)]_0^\pi = \cdots$$
$$= [T_n(x)P^{(n)}(x) + \cdots + \sin(x)P'(x) - \cos(x)P(x)]_0^\pi,$$

where $T_n(x)$ is one of the four functions: $\pm\sin(x), \pm\cos(x)$. The last expression is clearly an integer as a sum of integers.

Now suppose that $\pi = \frac{m}{n}$, where m and n are positive integers. Then $m - 2n\pi = m - 2n\frac{m}{n} = -m$, so that the polynomial $Q(x) = m - 2nx$ is in $\mathcal{P}$. Further define for each $k \in \mathbb{N}$ a polynomial Q_k as follows:

$$Q_k(x) = \frac{1}{k!}x^k(m - nx)^k.$$

Then

$$Q'_{k+1}(x) = \frac{1}{(k+1)!}((k+1)x^k(m-nx)^{k+1} + x^{k+1}(k+1)(-n)(m-nx)^k)$$
$$= \frac{1}{k!}x^k(m-nx)^k(m-nx-nx) = Q_k(x)Q(x).$$

The last formula allows to deduce inductively that $Q_k(x)$ is in $\mathcal{P}$. Indeed, $Q_1(0) = Q_1(\pi) = 0$ and $Q'_1(x) = Q(x)$, so $Q_1(x) \in \mathcal{P}$. Now suppose that we have already established that $Q_k(x) \in \mathcal{P}$. From the formula above and property (i) of the family $\mathcal{P}$ it follows that $Q'_{k+1}(x) \in \mathcal{P}$.

As $Q_{k+1}(0) = Q_{k+1}(\pi) = 0$, we actually get $Q_{k+1}(x) \in \mathcal{P}$ and the inductive reasoning is finished.

Note further that each $Q_k(x) > 0$ for all $x \in (0, \pi)$, as for such x we have $m - nx > 0$. This implies that $\int_0^\pi \sin(x)Q_k(x)\mathrm{d}x > 0$. But as $Q_k(x)$ is in $\mathcal{P}$, property (ii) above implies that the last integral is integer, so that in particular

$$\int_0^\pi \sin(x)Q_k(x)\mathrm{d}x \geq 1. \qquad (1)$$

Denote by M the maximum of the function $x(m - nx)$ on the interval $[0, \pi]$ (it is obviously finite and can be computed explicitly). Now the standard estimate implies that

$$\int_0^\pi \sin(x)Q_k(x)\mathrm{d}x \leq \int_0^\pi 1\frac{1}{k!}M^k\mathrm{d}x = \pi\frac{M^k}{k!}. \qquad (2)$$

Comparison of (1) and (2) gives

$$\pi\frac{M^k}{k!} \geq 1, \quad k \in \mathbb{N}.$$

This is contradictory: it is easy to show that $\pi\frac{M^k}{k!} \stackrel{k\to\infty}{\longrightarrow} 0$, so it cannot remain greater than 1 for ever . . . . We have reached the statement we were after.

Statement 3. *π is irrational.*

The diligent reader will easily show that the reasoning given above can be modified to establish the following result of A Parks [2]:

Theorem 4. *Suppose that $c > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = f(c) = 0$, $f(x) > 0$ for all $x \in (0, c)$. Suppose further that there exists a sequence of functions $(f_k)_{k=1}^\infty$ such that $f_1' = f$, $f_{k+1}' = f_k$ and $f_k(0) \in \mathbb{Z}$, $f_k(c) \in \mathbb{Z}$ for all $k \in \mathbb{N}$. Then c is irrational.*

It may seem technical and not very interesting, but it actually shows that

- if $0 < |r| \leq \pi$ and both $\sin(r), \cos(r)$ are rational, then r is irrational;
- if $r > 0$ is rational and not equal to 1 then $\ln r$ is irrational.

Try to find in both cases the functions f needed to apply Theorem 4!

π is irrational.

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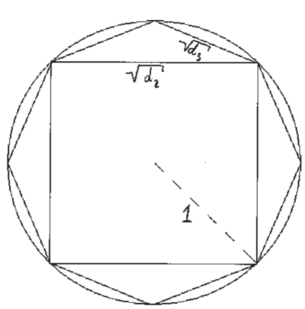
Niven's proof was deemed to be beautiful enough to be considered 'a proof from *The Book*' and as such found its way to the well-known (and wholeheartedly recommended) volume [3]. The phrase used above was coined by P Erdös who half seriously claimed that God has a book (*The Book*!) containing most striking, elegant and important mathematical reasonings. Presumably Erdös believed that every result has an 'ultimate' proof; although this may be questioned in many cases, it is difficult to imagine that there may be a simpler or more elementary proof of the fact that π is irrational than the one discovered by I Niven and reproduced earlier.

Is π the same type of an irrational number as $\sqrt{2}$? It turns out that they are actually very different: the latter is defined as a unique nonnegative solution of the equation $x^2 - 2 = 0$, and so, in particular, falls into the class of *algebraic* numbers, that is those which are solutions of polynomial equations with rational coefficients. In 1882 a German mathematician F Lindemann showed using analytic functions that π is *transcendental* (i.e. not algebraic). Lindemann's proof was later simplified several times and may be found in an accessible form in Chapter 12 of the book [4].

We have shown that the decimal expansion of π does not contain any periodic 'pattern'; can we actually produce decent approximations using basic principles? Look at a circle C with radius 1 and consider regular polygons A_k with 2^k vertices inscribed in C. Using elementary geometric principles and induction on k, one can show that the perimeters of A_k (denoted further by p_k) tend to the circumference of C as k tends to infinity and deduce the following recurrent relation:

Figure 2.



$$p_k = 2^k \sqrt{\mathrm{d}_k}, \quad \text{where} \quad \mathrm{d}_{k+1} = 2 - \sqrt{4 - \mathrm{d}_k}.$$

This leads to the following limit expression:

$$\pi = \lim_{k\to\infty} 2^{k-1} \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}},$$

where the braced expression contains $k-2$ terms. This was already known to a French mathematician Vieta in the 16th century, but it is still a pleasure to conduct the whole argument oneself, and I advise the reader to do so. Further exploiting the recurrence above one can write a short computer program computing consecutive values p_k and see how quickly does the sequence above converge and to what extent the approximations agree with the decimal expansion of π. Note that one can actually estimate the difference $2\pi - p_k$.

Another easy but instructive technique of finding approximate value of π using a computer is based on *the Monte Carlo method*.

Another easy but instructive technique of finding approximate value of π using a computer is based on *the Monte Carlo method*: If we can generate randomly (with the uniform distribution) pairs of numbers in the interval $[-1,1]$, we can view them as points in the square $[-1,1] \times [-1,1]$ and expect that on average $\frac{\pi}{4}$ of them should fall into the disk of radius 1 centered in $(0,0)$. Strong law of large numbers tells us that if we keep producing bigger and bigger samples of such numbers the actual proportion of points inside the disk should converge to $\frac{\pi}{4}$. Obviously 'random' numbers generated by the computer cannot be rigorously guaranteed to be uniformly distributed (as they are only 'pseudo-random') and, even if they were, in comparison to the last method one cannot easily estimate the error of the obtained approximation.

There is another very interesting 'probabilistic' aspect of the decimal expansion of π. A number is called *simply normal* if it has an infinite decimal expansion and the ratio of occurrences of any given digit in this expansion tends to $\frac{1}{10}$ (thus for example in the first million of digits there should be approximately 100000 zeros). It is conjectured that π might be even *normal*, that is any given finite sequence of digits ultimately appears in the expansion with equal probability among all the sequences of the same length (thus for example the ratio of occurrences of 0567 in the decimal expansion divided into four digit blocks should tend to $\frac{1}{10000}$).

In the decimal expansion of π, is there a place where a thousand consecutive digits are all zero?

Although the statistical evidence based on the known billions of digits of the expansion of π seems to support the last claim, in fact even the answer to the following much easier question asked by a Dutch mathematician L Brouwer, is not known: 'In the decimal expansion of π, is there a place where a thousand consecutive digits are all zero?' Note that we are getting close to our original topic of 'patterns' that may be found in π.

Current methods of computing the digits in the decimal expansion of π (yes, there are still people who are doing it, or rather making computers do it in a more and more efficient way) are based on the algebraic number theory, in particular the theory of *modular functions*. The first steps in this direction, as with so many aspects of the modern number theory, were taken by the great S Ramanujan, who, in his works from the beginning of the 20th century, gave several expressions for π in terms of infinite series. Let us present one of them:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{(k!)4} \frac{1103 + 26390k}{396^{4k}}.$$

Although it looks rather daunting, in the last twenty-thirty years it turned out that the formulas of that type are in fact very useful. Firstly, they can be used to provide recurrent formulas for constructing approximations (in the same spirit as the Vieta method discussed before), and secondly, the resulting sequences of approximations converge very quickly. The expression above in particular can be used to provide a two-step recurrent algorithm approximating $\frac{1}{\pi}$ *quartically* – this roughly means that every iteration quadruples the number of correct digits of $\frac{1}{\pi}$ (in this particular case one starts with the first approximation to $\frac{1}{\pi}$ equal to $6 - 4\sqrt{2}$)! Note that the existence of the super-efficient algorithms as the one just described does not render the problem of computing consecutive digits trivial – recurrent steps require conducting operations such as multiplication or division on very long numbers, storing the results in the computer memory and careful estimating of the possible

The Leibniz Formula actually allows to show that although π is irrational, it may be expressed as a continued fraction:

$$\frac{\pi}{4} = \cfrac{4}{1 + \cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cdots}}}}} \;.$$

On the other hand, it is closely related to the celebrated Wallis Formula which presents π as an infinite product:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}.$$

Figure 3.

errors. This has made it become nowadays an intricate problem lying at the intersection of mathematics and computer science. The current (at the time of writing) record belongs to a Japanese group lead by Y Kanada which has obtained over 10^{12} digits in the expansion.

There are many known infinite series expansions for π. The simplest one is the Leibniz Formula:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

The easiest modern way to obtain it is via the Taylor series expansion of the arctan function around 0:

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^n, \quad |x| \leq 1.$$

It is not at all efficient for computing the digits of π, as consecutive approximations approach the limit very slowly – again I recommend to compute the sum of the first few (or few hundred) terms and compare it with the actual value. Leibniz formula was discovered and published in 1673-4 by a German mathematician G W Leibniz, one of the fathers of the differential calculus.

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It was independently obtained more or less at the same time by a Scotsman J Gregory and, two centuries earlier, by an Indian mathematician K G Nilakantha. For a good description of the fascinating history of these discoveries, their geometric inspirations related to astronomy, optics and cartography and also connections with the beginnings of the development of the modern theory of infinite series refer to a well-written article [5]. Nowadays, the Leibniz formula is viewed as one of the big family of the BBP-type formulas (BBP stands for D Bailey, P Borwein and S Plouffe); some of them were used recently to construct algorithms allowing to compute an arbitrary digit in the decimal expansion of π without knowing all the digits preceding it!

Despite having proved the fact that π is irrational, we have excluded the possibility of finding simple patterns in the decimal expansion, I hope that the discussions above help to convince the reader that there is still a lot of room for related investigations, many questions to be asked and undoubtedly many exciting discoveries on the way. Most of the mathematicians would agree with the statements of Max Cohen with which this article began, although each would probably offer his or her own interpretation to them. The quest for 'patterns', which may be seen as the quest for understanding has always been an important part of science in general, and of mathematics in particular. It may not promise 'the key to everything', but still offers a lot of joy and satisfaction.

The bibliography below contains all the references I used while writing this article; I especially recommend [6], a collection of crucial research papers in the history of investigating π and interesting survey articles. I would like also to thank Professor B Sury from the ISI Bangalore for helpful comments.

Suggested Reading

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