Complex Numbers and Plane Geometry*

Anant R Shastri

The representation of complex numbers as points of the Euclidean plane naturally leads to a two-way interaction between geometry and numbers. The geometry of the plane has a very deep influence in the study complex analytic functions. In this article, we illustrate the other way aspect by a few simple-minded application of complex numbers to give elegant solutions of problems in plane geometry, such as Ptolemy's Theorem, Euler-line and Nine-point Circle Theorem.

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1. Introduction

We begin with a story taken from the famous book *One Two Three...Infininity* by G Gamow: There was a young and adventurous man who found among his great-grandfather's papers a piece of parchment that revealed the location of a hidden treasure. The instructions read:

" Sail to ... North latitude and ... West longitude where thou wilt find a deserted island. There lieth a large meadow, not pent, on the north shore of the island where standeth a lonely oak and a lonely pine. There thou wilt see also an old gallows on which we once were wont to hang traitors. Start thou from the gallows and walk to the oak counting thy steps. At the oak thou must turn right *by a right angle and take the same number of steps. Put here a spike in the ground. Now must thou return to the gallows and walk to the pine counting thy steps. At the pine thou must turn* left *by a right angle and see that thou takest the same number of steps, and put another spike into the ground. Dig half-way between the spikes; the treasure is there."*

The story is that when the young man finally landed on

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the island, even though he could find the meadow and the two trees as described, the old gallows had totally disappeared! The story ends with the adventurous man returning empty-handed after some desperate digging at random.

Now, the point Gamow wants to make is that if the young man knew a bit of mathematics, particularly the use of complex numbers, he could have found the treasure. We only add that, even if the man had not despaired and just made a single attempt to guess the location of the gallows and from there onward, carried out the instructions given in the parchment, he would have got the treasure as well as the satisfaction (perhaps falsely) of guessing the position of the gallows correctly. Try to figure it out yourself before reading the solution so that you will have the satisfaction of finding the treasure.

I would like to emphasis the fact that, it is not very difficult to solve this problem through elementary school geometry, either. However, in this article, we shall see a very neat solution to this problem, using complex numbers. The idea behind this can also be used to solve many plane geometry problems as well. As an illustration, we shall prove Ptolemy's Theorem and Nine-Point Circle Theorem.

At this point, here is a question which does not need any knowledge of complex numbers nor any mathematics. Try to answer it to yourself:

Which side of the line from pine tree to oak tree lies the treasure?

We assume that the reader has some familiarity with basic operations of addition and multiplication of complex numbers. However we shall briefly recall them in Sections 2 and 3 and then present a solution of Gamow's problem in Section 4. In Section 5, we shall recall some more properties of complex numbers though not everything recalled is put to use immediately. The last three sections are devoted to discussion of plane geometry problems using complex numbers. The choice of these problems is purely a matter of taste.

2. Basics of Complex Numbers

A complex number is an expression of the form $x + \imath y$ where x and y are real numbers. It is important to understand that two complex numbers

$$z_1 = x_1 + \imath y_1; \quad z_2 = x_2 + \imath y_2$$

are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. For a complex number $z = x + \imath y$, x is called the *real part* of z and y is called the *imaginary part* of z. A real number r is identified with a complex number whose real part is r and the imaginary part is 0. The addition and multiplication of any two complex numbers is defined by:

$$z_1 + z_2 := (x_1 + x_2) + \imath(y_1 + y_2);$$

$$z_1 z_2 := (x_1 x_2 - y_1 y_2) + \imath(x_1 y_2 - y_1 x_2).$$

All the standard laws of arithmetic which hold for real numbers hold for complex numbers as well. Note that there are two very special complex numbers viz., $\pm\imath$ such that

$$(\pm\imath)^2 = -1.$$

The symbol $\imath$ is the Greek letter 'iota' and is pronounced 'eye'.

Since the real and imaginary parts of a complex number determine the complex number uniquely, it follows that the set of all complex numbers $z = x + \imath y$ is in a one-to-one correspondence with the ordered pairs (x, y) of real numbers x, y. To each complex number, $z = x + \imath y$, we can assign a point viz., (x, y) in the 2-dimensional coordinate plane. Of course, we should choose the origin

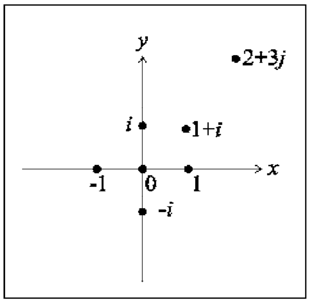


Figure 1. Argand diagram.

and the two perpendicular axes beforehand for this to make sense. It should also be noted that it is a commonly accepted practice to choose the two axes in such a way that the direction of moving from the positive x-axis to the positive y-axis is counterclockwise. Once the axes have been chosen, each point of the plane defines a unique complex number. At this stage we remark that the freedom in choosing the origin and the axes comes as a big bonus, as illustrated by various examples in this article.

The graphical representation of complex numbers is commonly known as the Argand diagram, (*Figure* 1), after the published work (1806) of Jean Robert Argand of Geneva, even though Gauss had used this idea in his thesis about eight years before.

The polar coordinate representation of points on the plane can also be exploited to represent complex numbers. Accordingly, each non-zero complex number z will have a unique expression $(r\cos\theta, r\sin\theta)$, where r is a positive real number equal to the distance of z from 0, and $0 \leq \theta < 2\pi$ is the angle subtended by the line segment $[0, z]$ with the positive real axis measured counterclockwise and in radians. Here r represents the distance of z from the origin. The angle θ is called the argument of the number z and denoted by arg z (*Figure* 2). The

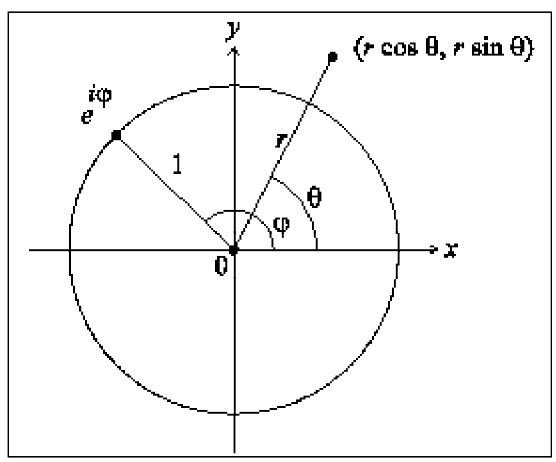


Figure 2. Polar coordinates.

number $\cos\theta + \imath\sin\theta$ is also denoted by $e^{\imath\theta}$. In this notation, we have $z = re^{\imath\theta}$, where $|z| = r$ and $arg\, z = \theta$. For the complex number 0 itself, we have $r = 0$ but arg 0 is not defined. Also the set of points z with $|z| = 1$ is the *unit circle* (*Figure* 2).

3. Geometric Way of Adding and Multiplying

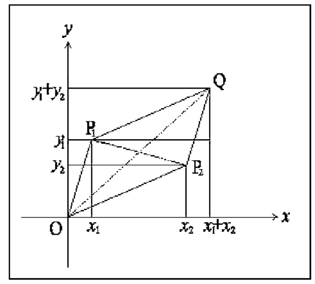


Figure 3. Addition of complex numbers.

The interaction between the geometry of the plane and the arithmetic of complex numbers begins as soon as we have represented the complex numbers by points in the plane. Let us see how to interpret addition and multiplication geometrically.

Start with two complex numbers z_1, z_2 which we want to add. If one of them is 0 there is not much to do. Similarly, if one is a real multiple of the other, say $z_2 = rz_1$ then $z_1 + z_2 = (1 + r)z_1$ and therefore, it essentially amounts to adding two real numbers. So, let us consider the most important case when z_1, z_2 are not a real multiple of each other. If P_1, P_2 are points represented by z_1, z_2 then this is the same as saying that O, P_1, P_2 are non-collinear respectively.

We then construct the parallelogram with O, P_1, P_2 as three of its vertices and OP_1, OP_2 as adjacent sides. The fourth vertex Q will then represent the sum $z_1 + z_2$. To prove this, you may drop perpendiculars on to the x and y axes from the points P_1, P_2 and Q and argue with some appropriate similar triangles (*Figure* 3).

Thus we see that in adding the number z_1 to the number z_2, we move the line segment $[0, z_1]$ parallel to itself, till the point corresponding to 0 coincides with the point z_2 and then just look at where the point z_1 itself has landed. Thus we can call the operation of adding z_1 as *parallel translation* by z_1.

In particular, if a line segment $[p, q]$ is parallelly translated so that the point p falls on the origin, then the point q will fall on the point representing the number $q - p$. This observation is the beginning of the so-called 'vector-method' in plane geometry. This is nothing very special about complex numbers. The true strength, as we shall see, comes from the geometric interpretation of multiplication.

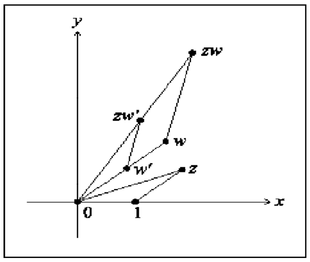


Figure 4. Multiplying complex numbers.

So, let us work out how to construct the point representing the product zw of two complex numbers z, w. Once again, we consider only the important case when $0, z, w$ are non-collinear, and z, w are both non-real. Consider the triangle with vertices $0, 1, z$. We construct another triangle $\Delta(0, w, q)$ which is similar to $\Delta(0, 1, z)$ so that the side $[0, 1]$ corresponds to the side $[0, w]$ and the two triangles are both labeled in the counterclockwise sense. Using similar triangles, it is not difficult to verify that the point q now represents zw, in a similar way as in the case of addition. We leave this as an exercise to the reader. However, if we use polar coordinates and a little bit of trigonometry, then the verification that $q = zw$ can be carried out as follows (*Figure* 4): Let

$$z = (r_1 \cos\theta_1, r_1 \sin\theta_1);$$

$$w = (r_2 \cos\theta_2, r_2 \sin\theta_2).$$

We rotate the triangle $(0, 1, z)$ through an angle θ_2 to obtain the triangle $\Delta(0, w', zw')$, say, where $w' = e^{\imath\theta_2}$. It follows that w' is a point on the line joining 0 and w and since this triangle is similar to $\Delta(0, w, q)$, it also follows that zw' will lie on the line joining 0 and q. Further, by similarity of triangles, it follows that the length of $[0, q]$ is equal to $r_2|zw| = r_1 r_2$. Therefore

$$q = r_1 r_2(\cos(\theta_1 + \theta_2), r_1 r_2 \sin(\theta_1 + \theta_2)).$$

Now use the formulae

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2;$$

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2,$$

to conclude that $q = zw$.

The advantage of this proof is that it gives us a complete geometric meaning of multiplication by a complex number $w = (r_2 \cos\theta_2, r_2 \sin\theta_2)$, viz., we rotate the line $[0, z_1]$ through an angle θ_2 and then expand/contract it by a factor r_2. In particular, if w is a positive real number then there is no rotation to be performed. If it is a negative real number, then we have to rotate by an angle π which corresponds to changing the sign. If the number w is at a unit distance from 0 then there is no scaling factor.

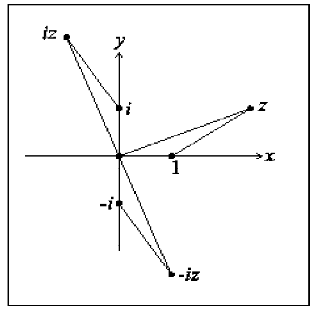


Figure 5. Multiplication by i and –i.

In particular, we see that multiplication by $\imath$ has the effect of rotating the vector through 90° in the counter-clockwise sense (*Figure* 5). Clearly multiplication by $-\imath$ amounts to rotation of the vector in the other direction through 90°.

4. Solution to Gamow's Treasure

Let us represent the map of the island by complex numbers. Of course, we are free to choose our axes and what is better than to choose the line joining the two trees as the real axis. Now, clearly, half-way between the two trees should be a good choice for the origin. Then it really should not matter whether the position of the oak or that of pine is chosen as the number 1 say, the pine. Then naturally the position of the oak will refer to -1. Now let Γ denote the position of the gallows, which is not known. The point is that *it does not matter: carry out the rest of the instructions and you arrive at an answer independent of this unknown quantity.* We feel that you should still try this problem on your own. At this stage we shall give you a hint – use the fact that multiplication by $\imath$ corresponds to turning a vector through a right angle in the anti-clockwise direction. Read *Box* 1 for the complete solution, only after you have tried enough to get the answer on your own.

Box 1.

The position S_1 of the first spike is found as follows: The vector representing the distance and the direction from the gallows to the oak is $-1-\Gamma$. Therefore, the vector representing the direction and the distance from the oak to the first spike is got by multiplying by $-\imath$, viz., $\imath(1+\Gamma)$. Since this vector has to originate at the oak, we see that $S_1 = \imath(1+\Gamma)-1$. Likewise, the position of the second spike is given by $S_2 = \imath(1-\Gamma)+1$. The midpoint of the line segment $[S_1, S_2]$ is then $\imath$ where the treasure is!

School Geometry Solution to Gamow's Problem: Draw perpendiculars S_1Q_1 and S_2Q_2 to the line joining O(ak) and P(ine). Using similar triangles show that the length of OP is equal to the sum of the lengths of these two perpendiculars. If TM is the perpendicular to OP from the midpoint T of S_1S_2 then M bisects OP and we have TM = OM = MP. (In particular, the position of T(reasure) is determined independently of the position of the gallows.)

5. More About Complex Numbers

Now that we have enough motivation to study complex numbers, let us quickly go through a few more basic properties of them.

For $z = x + \imath y$; we shall use the notation:

$$\Re(z) = x; \quad \Im(z) = y,$$

to denote the real and imaginary part of z respectively. The *complex conjugate* and the *modulus* of z are respectively defined by

$$\bar{z} = x - \imath y; \qquad |z| := (x^2 + y^2)^{1/2}.$$

Some of the important properties of these operations are listed below:

1. $|\bar{z}| = |z|; \quad z\bar{z} = |z|^2; \quad \overline{(\bar{z})} = z.$
2. $|z_1 z_2| = |z_1||z_2|.$
3. $|\Re(z) \leq |z|; |\Im(z)| \leq |z|.$
4. Cosine Rule: $|z_1 + z_2|^2 = |z_1^2| + |z_2|^2 + 2\Re(z_1\bar{z}_2).$

5. Parallelogram Law: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

6. Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

and equality holds if and only if one of the z_j is a non-negative real multiple of the other.

7. Cauchy's Inequality:

$$\left|\sum_{j=1}^{n} z_j w_j\right|^2 \leq \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right).$$

Properties 1,2,3, are easily verified. Property 4 is verified directly by expanding

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}.$$

Properties 5 and 6 are then easy consequences of 4. The last one is the only non-trivial one which can also be deduced from 4 inductively.

6. A Few Simple Geometric Problems

Now we shall consider a few geometric problems and seek answers to them using complex numbers, in the form of question and answer. They are arranged in such a way that the answer to one question will help to answer subsequent ones. You are encouraged to try each of them on your own before reading the answers.

Question 1: When are two line segments in the plane parallel to each other?

Question 2: When are three points A, B, C in the plane collinear?

Question 3: If two complex numbers z_1 and z_2 represent two points P_1 and P_2 respectively in the plane, how do you represent other points on the line passing through P_1 and P_2? What is the representation for the midpoint of the segment $[P_1, P_2]$?

Question 4: Show that the line segment joining the midpoints of two sides of a triangle is parallel to the third sid and is half its size.

Question 5: Show that the midpoints of the sides of a triangle form another triangle which is similar to the given one.

Question 6: Show that the midpoints of sides of any quadrilateral form a parallelogram.

Question 7: Given four points A,B,C,D in the plane such that AB $\perp$ BC and AB $\perp$ AD, show that the mid-point of [C,D] is equidistant from A and B.

Question 8: Show that in a rhombus the diagonals are perpendicular to each other.

Question 9: Given any quadrilateral, erect squares externally on each of the sides. Show that the centers of these squares form a quadrilateral whose diagonals are of equal length and perpendicular to each other.

Question 10: Given four distinct point in C, we define their *cross-ratio* to be the number given by

$$[z_1 : z_2 : z_3 : z_4] := \left(\frac{z_1 - z_3}{z_1 - z_4}\right) / \left(\frac{z_2 - z_3}{z_2 - z_4}\right).$$

Show that the four points lie on a circle or a straight line if and only if their cross ratio is a real number.

As an application of some of these simple observations, we shall give a refreshing proof of Ptolemy's theorem.

Ptolemy's Theorem: If A, B, C, D are vertices of a cyclic quadrilateral then

$$|\mathrm{AC}||\mathrm{BD}| = |\mathrm{AB}||\mathrm{CD}| + |\mathrm{AD}||\mathrm{BC}|.$$

Proof: If z_1, z_2, z_3, z_4 are the complex numbers representing A, B, C, D, listed cyclically, then it follows that $[z_1 : z_3 : z_2 : z_4] = \left(\frac{z_1 - z_2}{z_1 - z_4}\right) / \left(\frac{z_3 - z_2}{z_3 - z_4}\right)$ is a negative real number. That is $(z_1 - z_2)(z_3 - z_4) = r(z_1 - z_4)(z_3 - z_2)$, where $r < 0$. Therefore

$$\begin{aligned} & |(z_1 - z_2)(z_3 - z_4)| + |(z_1 - z_4)(z_3 - z_2)| \\ = & |(z_1 - z_2)(z_3 - z_4) - (z_1 - z_4)(z_3 - z_2)| \\ = & |(z_1 - z_3)(z_2 - z_4)| \end{aligned}$$

which yields Ptolemy's theorem. ♠

Now consider the following problem which can be solved in several ways. We find the solution using Ptolemy's theorem most elegant.

Example (*Figure* 6)

Suppose for $n \geq 4$, $A_1, A_2, \ldots, A_n$ be the vertices of a regular n-gon in that order such that

$$\frac{1}{|A_1A_2|} = \frac{1}{|A_1A_3|} + \frac{1}{|A_1A_4|}.$$

What is the value of n?

Solution: Put $t_j = |A_1A_j|, j = 2, 3, 4, 5$. Because $A_1A_2 \cdots A_n$ is a regular polygon.

We have

$t_2 = |A_2A_3| = |A_3A_4|, t_3 = |A_2A_4| = |A_3A_5|, t_4 = |A_2A_5|,$

etc.. Apply Ptolemy's theorem to the cyclic quadrilateral $A_1A_2A_3A_5$ to get

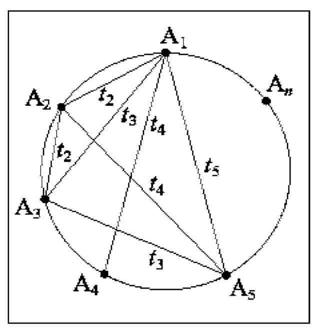
$$t_2t_3 + t_2t_5 = t_3t_4.$$

Also the given condition is equivalent to

$$t_3t_4 = t_2t_4 + t_2t_3.$$

Therefore $t_2t_5 = t_2t_4$ which means $t_4 = t_5$. This is possible if and only if $n = 7$.

Figure 6. Using Ptolemy's theorem.



Finally here are the answers to the questions that we raised earlier.

Answer 1. The segments AB and CD are parallel if and only if the two numbers A−B and C−D are real multiples of each other. This is the same as saying that the segments AB and CD are parallel if and only if $\frac{\mathrm{A}-\mathrm{B}}{\mathrm{C}-\mathrm{D}}$ is a real number.

Answer 2. Take A = D in the above question. We see that the points are collinear if and only if $\frac{\mathrm{A}-\mathrm{B}}{\mathrm{A}-\mathrm{C}}$ is a real number.

Answer 3. Any point on the line is represented by $(1-t)z_1 + tz_2$, where t ranges over real numbers. For $0 \le t \le 1$, we get all points inside the line segment $[\mathrm{P}_1, \mathrm{P}_2]$.

The midpoint corresponds to the value $t = 1/2$, i.e., $\frac{z_1+z_2}{2}$.

Answer 4. Represent the vertices of the triangle by three complex numbers z_1, z_2, z_3. Say the segment $[z_2, z_3]$ corresponds to the base. Then the midpoints of the two legs are given by $\frac{z_1+z_2}{2}, \frac{z_1+z_3}{2}$. Therefore, the length of the line segment joining them is equal to

$$\left|\frac{z_1+z_2}{2} - \frac{z_1+z_3}{2}\right| = \frac{|z_2-z_3|}{2}.$$

That this segment is parallel to the base follows from the previous exercise.

Answer 5. Follows from 4.

Answer 6. Easy.

Answer 7. Here we follow a method similar to the one followed in treasure hunt. Choose the midpoint of AB as the origin, A = −1 and B = 1. With respect to this coordinate choice, it follows that C $= 1 + \imath y_1$; D $= -1 + \imath y_2$. Therefore the midpoint of [C,D] is given by $\imath(y_1 + y_2)/2$ which is clearly equidistant from A $= -1$ and B $= 1$.

Answer 8. (*Figure* 7) Here $|z| = |w|$. Therefore

$(z+w)\overline{\imath(z-w)} = -\imath z\bar{z} + \imath w\bar{w} - \imath(-z\bar{w} + w\bar{z}) = 2\Im(w\bar{z})$, which is a real number.

Therefore $\imath(z - w)$ is parallel to $z + w$ which is the same as saying that $z - w$ is perpendicular to $z + w$.

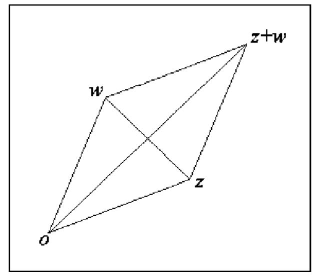


Figure 7. Diagonals of a rhombus are perpendicular.

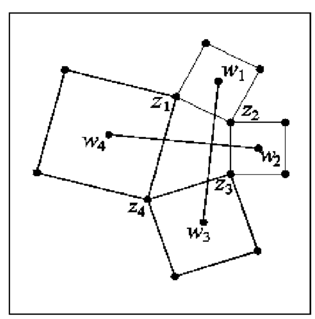
Answer 9. (*Figure* 8) Let $w_j, j = 1, 2, 3, 4$, be the respective centres of the squares. It suffices to show that $w_1 - w_4 = \pm\imath(w_2 - w_3)$. The midpoint of the segment $[z_1, z_2]$ is given by $(z_1 + z_2)/2$. The line perpendicular to this and of half its length is $[0, \imath(z_2 - z_1)/2]$. Therefore, it follows that $2w_1 = (z_1 + z_2) + (z_2 - z_1)\imath$. Likewise we have, $2w_2 = (z_2 + z_3) + (z_3 - z_2)\imath$; $2w_3 = (z_3 + z_4) + (z_4 - z_3)\imath$; and $2w_4 = (z_4 + z_1) + (z_1 - z_4)\imath$.

Observe that $(1 + \imath)\imath = \imath - 1$ and $(1 - \imath)\imath = 1 + \imath$.

Therefore, $2(w_3 - w_1) = (z_3 - z_1)(1 - \imath) + (z_4 - z_2)(1 + \imath)$; and $2(w_4 - w_2) = (z_4 - z_2)(1 - \imath) + (z_1 - z_3)(1 + \imath)$. Thus, we see that $(w_3 - w_1) = (w_4 - w_2)\imath$ as required.

Answer 10: Observe that the cross-ratio of four points depends upon the order in which we take them. However, check that if one of them is real then all are real too. If any three of them are collinear, say z_2, z_3, z_4 then we know that $\frac{z_2 - z_3}{z_2 - z_4}$ is a real number. But then z_1 lies on the same line if and only if $\frac{z_1 - z_3}{z_1 - z_4}$ is real, i.e., if and only if $[z_1 : z_2 : z_3 : z_4]$ is real.

Figure 8. External squares on a quadilateral.



Consider the case when no three are collinear. Let C be the circle through three of them say z_2, z_3, z_4. Then we know that a point z in the plane lies on C if and only if the two angles $\angle z_3 z z_4$ and $\angle z_3 z_2 z_4$ are equal or add up to π. This is the same as saying that the cross ratio $[z : z_2 : z_3 : z_4]$ is real. It will be positive or negative according as z, z_2 are on the same or opposite side of the chord $[z_3, z_4]$.

7. The Nine-Point Circle

We shall now present a proof of the celebrated nine-point circle theorem:

Theorem A: The circle passing through the midpoints of the sides of a triangle passes through the feet of the altitudes and the midpoints of the line segments joining the orthocentre to the vertices.

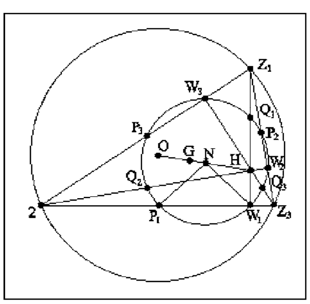
Thus given any triangle, nine of the geometrically meaningful points associated to the triangle are all found on a single circle. That circle deserves to be called the nine-point circle associated to the triangle. Of course for special triangles some of these points themselves may overlap, the worst case being that of an equilateral triangle.

We will use most of the results that we have discussed so far, in proving Theorem A. In fact, along the way, we shall also prove another important theorem due to Euler.

Theorem B (Euler Line): (*Figure* 9) The circumcentre O, the centre of gravity G, the centre N of the nine-point circle and the orthocentre H of a triangle are all collinear. Moreover, we have

$$\text{OG:ON:OH} = 2:3:6.$$

Figure 9. The nine-point circle and the Euler line.



Proof of Theorem B: We choose the orthocentre O of the given triangle to be the origin of the plane. Let the three vertices z_1, z_2, z_3 of the triangle be represented by the complex numbers z_1, z_2, z_3. Then it follows that

$$|z_1| = |z_2| = |z_3| =: r.$$

Using the fact that the diagonals of a rhombus are perpendicular to each other, (see Q.7 in the previous section), we conclude that

$$(z_i - z_j) \perp (z_i + z_j).$$

By definition, *the centroid* is given by

$$\mathrm{G} := \frac{z_1 + z_2 + z_3}{3}.$$

We claim that the orthocentre H is given by

$$\mathrm{H} = z_1 + z_2 + z_3.$$

For, the orthocentre is the unique point which satisfies the property

$$\mathrm{H} - z_1 \perp (z_2 - z_3);\ \ \mathrm{H} - z_2 \perp (z_3 - z_1);\ \ \mathrm{H} - z_3 \perp (z_1 - z_2).$$

And it is easily verified that $\mathrm{H} = z_1 + z_2 + z_3$ satisfies these conditions.

Thus we have proved that O, G and H are collinear and also that

$$\mathrm{OG{:}OH} = 1{:}3.$$

In the literature, OGH is called the *Euler line.* Let now

$$\mathrm{P}_1 = \frac{z_2 + z_3}{2};\ \mathrm{P}_2 = \frac{z_3 + z_1}{2};\ \mathrm{P}_3 = \frac{z_1 + z_2}{2}$$

be the midpoints of the three sides of the given triangle. Let N be the centre of the circle C passing through $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$. Thus N is the unique point which is equidistant from $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$. We verify that $\frac{\mathrm{H}}{2} = \frac{z_1 + z_2 + z_3}{2}$ has this property, i.e.,

$$\left|\frac{\mathrm{H}}{2} - \mathrm{P}_i\right| = \frac{|z_i|}{2} = \frac{r}{2}.$$

Thus, $\mathrm{N} = \mathrm{H}/2$. This completes the proof of Theorem B. ♠

We now turn our attention to Theorem A. Let us first prove that the point Q_j which is midpoint of HZ_j is on the circle C for each $j = 1, 2, 3$.

But then

$$Q_j = \frac{H + z_j}{2}.$$

Therefore,

$$|N - Q_j| = \left| -\frac{z_j}{2} \right| = \frac{r}{2}.$$

(In fact, $N - Q_j = -(N - P_j)$ shows that Q_j are antipodal to P_j with respect to N).

We shall now show that W_j are also on C.

For this, we observe that HW_1 and OP_1 are both perpendicular to the line $z_2 z_3$. And N is the midpoint of the line OH. Therefore, as seen in Question 6, N is equidistant from P_1 and W_1. Therefore W_1 is on C. Likewise W_2 and W_3 are also on C. This completes the proof of Theorem A. ♠

8. Some More Geometric Problems

The reader is welcome to try each one of the following questions. We advise that only after trying enough or after obtaining a solution on one's own, one may read the solutions given below which are based on the use of complex numbers. Indeed, many of these problems can be solved by somewhat different methods. However, the only solution I know of Problem 7 is via Problem 6.

[Q.1] Show that three points represented by z_1, z_2, z_3 form an equilateral triangle if and only if $z_2 - z_1 = \omega(z_2 - z_3)$, where $\omega = e^{\pm \pi \imath/3} = \frac{-1 \pm i\sqrt{3}}{2}$.

[Q.2] If $z_1 + z_2 + z_3 = 0$, $|z_j| = 1$, then show that $\{z_1, z_2, z_3\}$ forms the vertex set of an equilateral triangle.

[Q.3] Show that the centers of the equilateral triangles erected externally on the three sides of a given triangle form an equilateral triangle.

[Q.4] Suppose $w_1, w_2, w_3 \in \mathrm{C}$ represent three non-collinear points and $w_1 + w_2 + w_3 = 0$. If a_1, a_2, a_3 are real numbers such that $a_1w_1 + a_2w_2 + a_3w_3 = 0$, then show that $a_1 = a_2 = a_3$. (This is equivalent to say that if three planar forces acting on a point are keeping it in equilibrium then by scaling all the three forces by the same factor and only by the same factor, the point will still be in equilibrium.)

[Q.5] Let $w_1, w_2, w_3 \in \mathrm{C}\setminus\{0\}$ be such that $w_1+w_2+w_3 = 0$. Prove that the following statements are equivalent.

(i) $w_1^2 + w_2^2 + w_3^2 = 0$.

(ii) $w_1w_2 + w_2w_3 + w_3w_1 = 0$.

(iii) $\dfrac{1}{w_1} + \dfrac{1}{w_2} + \dfrac{1}{w_3} = 0$.

(iv) $|w_1| = |w_2| = |w_3|$.

(v) $0, w_1, w_1 + w_2$ form the vertices of an equilateral triangle.

(vi) w_1, w_2, w_3 form the vertices of an equilateral triangle.

[Q.6] Prove that three distinct points z_1, z_2, z_3 in the plane form the vertices of an equilateral triangle if and only if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

[Q.7] Show that if w_1, w_2, w_3 are points dividing the three sides of the triangle $\Delta(z_1, z_2, z_3)$ in the same ratio, then the triangle $\Delta(w_1, w_2, w_3)$ is equilateral if and only if the triangle $\Delta(z_1, z_2, z_3)$ is equilateral.

Answers:

[A.1] The triangle is equilateral if and only if $|z_2 - z_1| = |z_2 - z_3|$ and the angle between the two segments is equal to $\pm\pi/3$.

[A.2] Consider the isosceles triangle $z_1, z_1 + z_2, 0$. The given condition implies that it is equilateral. Therefore the angle between z_j and z_k must be $2\pi/3$. This means $z_2 = z_1\omega, z_3 = z_1\omega^2$ where $\omega \neq 1$ is a cube root of unity. Observe that $1, \omega, \omega^2$ form the vertex set of an equilateral triangle. Therefore, z_1, z_2, z_3 also form an equilateral triangle.

[A.3] Let z_1, z_2, z_3 denote the vertices of the given triangle labeled clockwise. The center w_1 of the equilateral triangle raised on the side $[z_1, z_2]$ is such that if $\mathrm{M}_1 = \frac{z_1+z_2}{2}$ is the midpoint of $[z_1, z_2]$ then $[\mathrm{M}_1, w]$ is perpendicular to the side $[z_1, z_2]$ and is of length $\dfrac{1}{2\sqrt{3}}$ times the length of the side $[z_1, z_2]$. Therefore

$$w_1 = \frac{z_1 + z_2}{2} + \frac{\imath(z_2 - z_1)}{2\sqrt{3}}.$$

Likewise,

$$w_2 = \frac{z_2 + z_3}{2} + \frac{\imath(z_3 - z_2)}{2\sqrt{3}}; \quad w_3 = \frac{z_3 + z_1}{2} + \frac{\imath(z_1 - z_3)}{2\sqrt{3}}.$$

Verify that $w_2 - w_1 = \omega(w_3 - w_1)$,
where $\omega = e^{\pi\imath/3} = \dfrac{1 + \imath\sqrt{3}}{2}$.

[A.4] We have

$$w_1 + w_2 + w_3 = 0 = a_1w_1 + a_2w_2 + a_3w_3.$$

Multiply the first one by $a_j, j = 1, 2$ and subtract from the second to get

$$(a_2-a_1)w_2+(a_3-a_1)w_3 = 0 = (a_1-a_2)w_1+(a_3-a_2)w_3.$$

Observe that none of the points w_j is 0. Therefore, if both these equations were nontrivial, then together they imply that w_1, w_2, w_3 are collinear. We conclude that one of the equations is trivial, say the first one, i.e., $(a_2 - a_1) = (a_3 - a_1) = 0$.

[A.5] (i) $\Leftrightarrow$(ii)

$0 = (w_1+w_2+w_3)^2 = w_1^2+w_2^2+w_3^2-2(w_1w_2+w_2w_3+w_3w_1).$

(ii) $\Leftrightarrow$(iii) Multiply and divide by $w_1w_2w_3$.
(iii) $\Leftrightarrow$ (iv) $w_j^{-1} = \bar{w}_j/|w_j|^2$. We have $w_1 + w_2 + w_3 = 0$, i.e., $\bar{w}_1 + \bar{w}_2 + \bar{w}_3 = 0$. Therefore, we have

$$\frac{|w_1|^2}{w_1} + \frac{|w_2|^2}{w_2} + \frac{|w_3|^2}{w_3} = 0.$$

Now appeal to the previous exercise.

The equivalence of (iv) , (v) and (vi) is easy. (Remember $w_1 + w_2 + w_3 = 0$).

[A.6] Put $w_1 = z_2 - z_1, w_2 = z_3 - z_2, w_3 = z_1 - z_3$. Then z_1, z_2, z_3 form the vertex set of an equilateral triangle if and only if so do w_1, w_2, w_3. Now the given condition for z_j is equivalent to (ii) of the previous exercise.

[A.7] Put $w_1 = tz_1 + (1-t)z_2, w_2 = tz_2 + (1-t)z_3, w_3 = tz_3 + (1-t)z_1$. Then verify that

$$\sum w_1^2 - \sum w_1w_2 = [t^2-(1-t)^2-t(1-t)](\sum z_1^2 - \sum z_1z_2).$$

Since $t^2 + (1-t)^2 - t(1-t) > 0$ for all $t \in$ R, we are done.

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Suggested Reading

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