Solution to 'On a Result of Ramanujan'

S K Varma and C L Dewangan
Govt. Chattisgarh P G College
Raipur 492 001, India.

The following claim is made in Ramanujan's note book (page 23, entry 11):

$$\text{If } x = y + \left(\frac{z}{y} - y^3\right)^{1/3}, \quad \text{then} \quad 3y = x + \left(\frac{9z}{x} - x^3\right)^{1/3}. \tag{1}$$

Iyer [1] has shown that 3 and 9 cannot be replaced by any other pair of numbers.

Iyer [1] has also obtained a similar result for quadratics, i.e.,

$$\text{if } x = -y + \left(\frac{z}{y} + y^2\right)^{1/2}, \text{ then } 4y = -x + \left(\frac{8z}{x} + x^2\right)^{1/2} \tag{2}$$

and 4 and 8 cannot be replaced by any other pair of numbers. We shall show that the above results cannot be generalized in the following form.

$$\text{If } x = -y + \left(\frac{z}{y} + y^n\right)^{1/n}, \text{ then } ay = -x + \left(\frac{bz}{x} + x^n\right)^{1/n} \quad \text{for unique } a \text{ and } b. \tag{3}$$

TIO question appeared in *Resonance*, Vol.12, No.1, pp.80–81, 2007.

The problem was posed by
T S K V Iyer
201, Oxford Premium P2
Oxford Village, Wanowarie
Pune 400 041, India.

Proof: Let (3) hold. Then

$$(x+y)^n = \frac{z}{y} + y^n$$

or

$$x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_{n-1} x y^{n-1} + y^n = \frac{z}{y} + y^n$$

where $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$ for $r \geq 0$,

or

$$x^{n-1} + {}^nC_1 x^{n-2} y + {}^nC_2 x^{n-3} y^2 + \ldots + {}^nC_{n-1} y^{n-1} = \frac{z}{xy}. \tag{4}$$

Keywords
A result of Ramanujan.

Similarly expanding, $(ay+x)^n = (bz/x + x^n)$, we get

$$a^n y^n + nC_1 a^{n-1} y^{n-1} x +^n C_2 a^{n-2} y^{n-2} x^2 + \ldots +$$

$$^nC_{n-1} ayx^{n-1} + x^n = \frac{bz}{x} + x^n,$$

or

$$a^n y^{n-1} +^n C_1 a^{n-1} y^{n-2} x +^n C_2 a^{n-2} y^{n-3} x^2 + \ldots +$$

$$^nC_{n-1} ax^{n-1} = \frac{bz}{xy}. \tag{5}$$

So, if (3) is true, comparing coefficients in (4) and (5) we get,

$$\frac{a^n}{^nC_{n-1}} = \frac{^nC_1 a^{n-1}}{^nC_{n-2}} = \frac{^nC_2 a^{n-2}}{^nC_{n-3}} = \ldots = \frac{^nC_{n-1} a}{1} = \frac{b}{1}.$$

Now taking the case $n = 2$ we have

$$\frac{a^2}{2} = \frac{2a}{1} = \frac{b}{1} => a = 4 \text{ and } b = 8.$$

For $n = 3$, one has

$$\frac{a^3}{3} = \frac{3a^2}{3} = \frac{3a}{1} = \frac{b}{1} => a = 3 \text{ and } b = 9.$$

Thus the result of Iyer [1] is obtained. But for $n = 4$,

$$\frac{a^4}{4} = \frac{4a^3}{6} = \frac{6a^2}{4} = \frac{4a}{1} = b.$$

This set of equations is inconsistent.

Similarly $n = 5$ gives

$$\frac{a^5}{5} = \frac{5a^4}{10} = \frac{10a^3}{10} = \frac{10a^2}{5} = \frac{5a}{1} = \frac{b}{1}.$$

This set of equations is also inconsistent.

Similarly for $n = 6, 7, \ldots$ a unique value of a cannot be obtained. For $n \geq 6$, the reader may supply the proof in a similar way.

The following results can also be proved,

If $x = -y + \left(\frac{z}{y} + y^3\right)^{1/3}$, then $3y = -x + \left(\frac{9z}{x} + x^3\right)^{1/3}$.

Here 3 and 9 cannot be replaced by any other pair of numbers.

And if

$$x = y + \left(\frac{z}{y} + y^2\right)^{1/2} \text{ then } 4y = x + \left(\frac{-8z}{x} + x^2\right)^{1/2}.$$

Again, 4 and 8 cannot be replaced by any other pair of numbers.

Suggested Reading

[1] **T K S V Iyer, On a Result of Ramanujan, *Resonance*, Vol.12, No.1, pp.80–81, 2007.**

Errata

***Resonance*, Vol.12, No.10, pp.67–70, 2007.**

Classroom:

Root Test and Ratio Test in the Context of (C, k) Summability of Series by P N Natarajan

$k \geq 0$ should read as $k > 0$ in the following lines :

Page 67: line 5 from top;
Page 68: line 5 from bottom;
Page 69: line 2 & line 3 from top;
Page 70: line 8 from top & line 6 from bottom

$k > -1$ should read as $0 > k > -1$ in the following lines:

Page 69: line 7 from bottom;
Page 70: line 3 from bottom. .