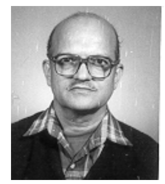
Seven Different Proofs of the Irrationality of $\sqrt{2}$

V G Tikekar

Atiyah (winner of Abel prize for 2004, jointly with Isador Singer) says[1] "Any good theorem should have several proofs, more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions – they are not just repetitions of each other". Following this remark by a great mathematician, we have given in this article, seven proofs for the irrationality of square root of 2.



V G Tikekar is a former professor of mathematics at IISc, Bangalore. His research interests are in mathematical programming, numerical mathematics, theoretical computer science, probability and statistics.

1. Introduction

Of all the concepts in Mathematics, the most important is that of a proof. One of the greatest mathematicians of all times, G H Hardy (1877–1947), the discoverer of the topmost Indian mathematical genius S Ramanujan, once told [1] another great mathematician and philosopher Bertrand Russel: "If I could prove by logic that you would die in five minutes, I should be sorry you were going to die, but my sorrow would be very much mitigated by pleasure in the proof". That brings out the place of 'proof' in the realm of mathematics, and shows how much significance we should attach to the proofs in our studies.

[1] See Notices of the *American Mathematical Society*, pp.225, February 2005.

Certain mathematical propositions (i.e., statements), especially the important ones, are proved in many different ways. For example, there is a collection [2] of 367 proofs of a single theorem that is famously known in the name of Pythagoras. These 367 proofs have been classified as algebraic, geometric, dynamic, or quaternionic by the author of this collection. It is instructive to collect all possible proofs of any particular proposition together,

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and compare them from the points of brevity, simplicity, elegance, general applicability etc.

2. Seven Proofs

To whet the appetite of the reader of such an exercise, we give here seven different proofs to prove that $\sqrt{2}$ is an irrational number. This proposition comes comparatively at an early stage in the mathematical preparation of a student of mathematics. There are some more proofs which prove that $\sqrt{2}$ is irrational.

***Proof* 1.** Let us assume contrary to what is to be proved. Thus we assume that $\sqrt{2}$ is not an irrational number. In other words, we take it that $\sqrt{2}$ is rational. So $\sqrt{2}$ being rational, we can express it in the form p/q, where p and q are integers, and $q \neq 0$. Let us, without loss of generality, assume further that p/q is reduced to its lowest terms, i.e., p and q have no common factor (other than 1), or symbolically $(p, q) = 1$. Then

$$(\sqrt{2} = p/q) \Longrightarrow (2 = p^2/q^2) \Longrightarrow [p^2(= 2q^2) \text{ is even}] \Longrightarrow (p \text{ is even}).$$

Therefore let $p = 2n$, n being integer. Hence

$$(4n^2 = 2q^2) \Longrightarrow [q^2(= 2n^2) \text{ is even}] \Longrightarrow (q \text{ is even}).$$

This proves that both p and q are even, i.e. they have common factor 2, a countradiction to our assumption that p and q have no common factors (other than 1). This implies that assuming $\sqrt{2}$ to be rational leads to contradiction and hence that assumption cannot be correct. So $\sqrt{2}$ is not rational, i.e., it is irrational. Q.E.D.

NOTE: The prerequisite for this proof is that (m is integer, and m^2 is even) $\Longrightarrow$ (m is even). It is easy to establish this assertion.

***Proof* 2.** We assume that $\sqrt{2}$ is a rational number. So by definition of a rational number, $\sqrt{2}$ can be expressed in the form p/q, where p and q are integers, and $q \neq 0$. Here $q > 1$ since $1 < \sqrt{2} < 2$ implying that $\sqrt{2}$ is a positive number but not an integer. Further, without loss of generality we assume that p and q have no common factors (other than 1). Therefore $2 = p^2/q^2$ and so

$$p^2/q = 2q. \qquad (1)$$

Now, q being integer, RHS of (1) is an integer, while LHS of (1) is not an integer since $q > 1$, and the numerator, namely $(p^2 = p \times p)$ and the denominator, viz. q have no common factor (other than 1). Thus we reach a contradiction that LHS and RHS of the equality (1) are of different forms. As the contradiction arises, our assumption cannot be granted. Therefore $\sqrt{2}$ is irrational. Q.E.D.

***Proof* 3.** Assume that $\sqrt{2}$ is not an irrational number. So we can express $\sqrt{2}$ as p/q, where p and q are positive integers, and $p > q > 1$ since $1 > \sqrt{2} < 2$. Assume further that p/q is in its lowest form (which can always be done). Now construct a square ABCD with side q (*Figure* 1).

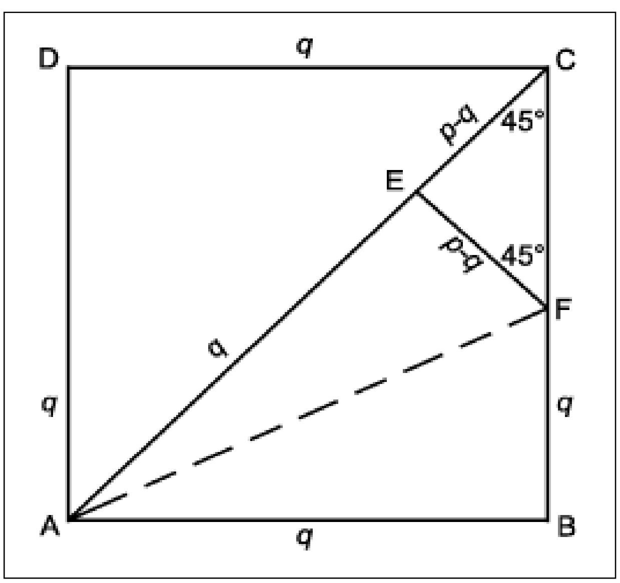

Figure 1.

Therefore

$$\begin{aligned} \text{AC} &= q\sqrt{2} \text{ (by Pythagoras theorem)} \\ &= q(p/q) \text{ (because of our assumption)} \end{aligned}$$

$$\therefore \text{AC} = p. \quad (2)$$

$$\text{Take AE} = q \text{ along the diagonal AC.} \quad (3)$$

(This is possible since AC$= q\sqrt{2} > q$). Construct EF $\perp$ AC with F on BC. (F will be on BC since AE = AB, $\angle$ EAB = 45° and $\angle$ AEB = 67.5°).

Using (2) and (3) we have

$$\text{EC} = \text{AC} - \text{AE} = p - q, \quad (4)$$

Now $\angle$ ACB = 45° since ABCD is a square. Therefore $\angle$ CFE = 45° since $\angle$ CEF = 90°. Hence using (4) we have

$$\text{EF} = \text{EC} = p - q \quad (5)$$

Δ AEF and Δ ABF are congruent right triangles (since AF is common, using (3) AE = AB = q, and $\angle$ AEF = $\angle$ ABF = 90°). Hence, using (5)

$$\begin{aligned} &\text{FB} = \text{EF} = p - q \\ &\therefore \ \text{FC} = \text{BC} - \text{FB} = q - (p - q) = 2q - p. \quad (6) \end{aligned}$$

Again since EF $\perp$ AC, using (5) we get Δ CEF as an isoscales right triangle. Therefore FC$= (p - q)\sqrt{2}$. Now using (6)

$$2q - p = (p - q)\sqrt{2} \quad \text{or} \quad \sqrt{2} = (2q - p)/(p - q).$$

Now since p, q are assumed to be integers, so are $2q - p$ and $p - q$. Further, since $1 < \sqrt{2} < 2$, we get $1 < p/q < 2$, or $q < p < 2q$ and hence $p - q > 0$, $p - q < q$, and $2q - p > 0$. Similarly, $q < p$ yields $2q - 2p < 0$, or $2q - p < p$.

We had assumed to start with that $\sqrt{2} = p/q$ is in its lowest terms, and $p, q > 0$. Now, we notice that $\sqrt{2} =$

$(2q-p)/(p-q)$, with $2q-p > 0$, $p-q > 0$, and $2q-p < p$, $p-q < q$. Thus, we now have a smaller numerator as well as a smaller denominator in the new expression for $\sqrt{2}$, i.e., we have been able to reduce the fractional form for $\sqrt{2}$ beyond what was assumed to have been in its lowest terms, a reduction that would be patently impossible. As the contradiction has been reached, our starting assumption, namely $\sqrt{2}$ is not an irrational is a mistake. This establishes the irrationality of $\sqrt{2}$. Q.E.D

***Proof* 4.** We assume that $\sqrt{2}$ is not an irrational number. We can then express $\sqrt{2}$ as p/q, where p and q are integers, and $q \neq 0$. Therefore $2q^2 = p^2$. Consider the prime factors of both the sides of this equality. RHS is p^2, a square of the integer p. Therefore, in the prime factorisation of the RHS ($= p^2$), 2 must appear an even number of times. (Note that zero is even). Similarly, 2 must occur an even number of times in the prime factorisation of q^2. As the entire LHS of above equality is $2q^2$, the additional coefficient 2 in it makes the prime factorisation of the LHS contain 2, an odd number of times. Thus we have the contradiction that the prime factors of the same number (call it p^2 once, and $2q^2$, the other time as $p^2 = 2q^2$) contain an odd as well as even number of 2's. Same number cannot have different prime factorisation. Hence our assumption to start with, viz. $\sqrt{2}$ is not irrational is not acceptable. So $\sqrt{2}$ is irrational. Q.E.D

***Proof* 5.** Suppose $\sqrt{2}$ is not an irrational number. That means we are assuming $\sqrt{2}$ to be rational. So by the definition of rational number, we can express $\sqrt{2}$ as p/q, where p and q are integers, and $q \neq 0$. Let us further take that the fraction p/q is expressed in its lowest form (which can always be achieved). Then p and q have no common factor (other than 1). Since p and q have no common factor (other than 1), p^2 and q^2 also have no common factor (other than 1). That means p^2/q^2 is a fraction expressed in its lowest terms. But $p^2/q^2 = 2$.

So $q^2 = 1$, and hence $q = 1$. Therefore $p = \sqrt{2}$. Thus $\sqrt{2}$ is integral (as p is integer), which is readily seen to be false as $1^2 = 1, 2^2 = 4$ giving $1 < \sqrt{2} < 2$. We thus get a contradiction (namely, $\sqrt{2}$ is integral), making our starting assumption, viz., $\sqrt{2}$ is not irrational, false. Hence $\sqrt{2}$ is irrational. Q.E.D.

***Proof* 6.** Assume contrary to what is to be proved. That is, assume that $\sqrt{2}$ is rational. Therefore using a property of rational numbers, we can express $\sqrt{2}$ as

$$\sqrt{2} = p/q, \text{ where } p, q \text{ are integers, and } \; q \neq 0. \qquad (7)$$

Assume further, without loss of generality, that p/q is in its lowest form.

$$(7) \Longrightarrow p^2 = 2q^2. \qquad (8)$$

We also have

$$1 < \sqrt{2} < 2. \qquad (9)$$

Now (8) $\Longrightarrow (p^2 - pq = 2q^2 - pq) \Longrightarrow [p(p-q) = q(2q-p)] \Longrightarrow p/q = (2q-p)/(p-q)$.

Using (9) and (7), $2 = \sqrt{2}.\sqrt{2} < 2(p/q)$

$$\therefore 2q < 2p, \text{ or } q < p, \text{ or } p - q > 0, \text{ and } 2q - p < p. \quad (10)$$

Again using (7) and (9), $p/q = \sqrt{2} < 2$

$$\therefore 2q - p > 0, \text{ and } p - q < q. \qquad (11)$$

From (10) and (11) it is clear that we have been able to reduce the fractional form for $\sqrt{2}$, to $(2q-p)/(p-q)$ which is smaller beyond what was assumed to be in its lowest form (namely p/q). Thus we reach to a contradiction because of our starting assumption that $\sqrt{2}$ is rational.

Therefore our starting assumption is incorrect and so $\sqrt{2}$ is irrational. Q.E.D.

***Proof* 7.** To prove that $\sqrt{2}$ is irrational, assume that $\sqrt{2}$ is rational to start with. So $\sqrt{2}$ can be expressed as

$$\sqrt{2} = p/q, \text{ where } p \text{ and } q \text{ are integers, and } q \neq 0. \quad (12)$$

Assume further that p/q is in its lowest form (which can always be acheived).

p/q cannot be integer as $1 < \sqrt{2} < 2$. Now since p/q is in its lowest form, so is p^2/q^2. So

$$(12) \Longrightarrow (p^2/q^2 = 2) \Longrightarrow (q^2 = 1) \Longrightarrow (q = 1)$$

Therefore $p^2 = 2$, implying (since p is an integer) that 2 is a perfect square, which is impossible since $1 < \sqrt{2} < 2$. Thus we reach a contradiction (namely, 2 is a perfect square). Therefore our starting assumption is incorrect. So $\sqrt{2}$ is irrational. Q.E.D.

3. Comments

By comparing and evaluating these seven proofs, each establishing the same proposition, one should be able to draw the following conclusions:

(i) All the seven proofs are different, but each of them is a proof by contradiction (reductio-ad-absurdum). The proposition under our consideration here, namely '$\sqrt{2}$ is an irrational number' cannot be established by direct-proof method; one has to resort to proof by contradiction. Here one remembers G H Hardy again, who described the method of proof-by-contradiction [3] as "one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit; a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game".

(ii) Proof 3 is very lengthy compared to the other ones.

(iii) Proofs 2, 4, 5, and 7 are short and elegant; they radiate beauty, though beauty is based on subjective appeal.

(iv) In Proof 6, subtracting pq from both sides of (8) in order to derive new reduced form for p/q looks artificial.

(v) Proofs 1 and 3 are useful for establishing irrationality of $\sqrt{2}$ only, while Proofs 2, 4, 5, 6, and 7 have general applicability in that the underlying methodology or the style of argument can be used to establish the irrationality of $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$, ..., $\sqrt{p}$ (p being prime), $\sqrt{p_1.p_2 \ldots p_r}$, $p_1, p_2, \ldots, p_r$ being distinct primes.

The Proof 6 can be generalised to general $\sqrt{n}$ (n being prime or product of distinct primes), if (a) we notice that there exists a natural number x such that $x < \sqrt{n} < x + 1$ and take this pair of inequalities in place of (9), and then (b) proceed to write $p^2 - xpq = nq^2 - xpq$ instead of $p^2 - pq = 2q^2 - pq$.

(vi) All the proofs, except Proof 4, assume that the representation of $\sqrt{2}$ in the form p/q is in its lowest form, but proof 4 does not require that assumption, i.e., in proof 4, p and q could have a common factor.

(vii) Proof 6 looks to be an algebraic manifestation of the geometric approach in Proof 3.

(viii) Proofs 2, 4, 5, 7 are based on expressing the starting equality $p^2/q^2 = 2$ as $p^2/q = 2q$, $p^2 = 2q^2$, $p = \sqrt{2}$ (noting that $q = 1$), and $p^2 = 2$ (noting that $q = 1$) respectively, and noting the contradiction each of them leads to. It is but natural that the starting assumption being incorrect, it can lead to various contradictions and depending on the approach adopted in a particular proof, different contradiction comes to surface.

4. Remarks

It is hoped that the reader is encouraged to collect different proofs of the same proposition to weigh them for different characteristics. Further, it is recommended that the reader studies the merits of different types of proofs [direct, indirect, reductio-ad-absurdum, proof by contradiction, constructive, existential, exhaustive, proof using mathematical induction (this also has two varieties)]. Such a study will help him/her appreciate mathematics more, and will improve his/her depth of understanding. Especially one must experience the joy and happiness embodied in the method of proof-by-contradiction, wherein a mathematician is ready to offer the sacrifice of the entire game, as aptly pointed out by Hardy.

Suggested Reading

[1] W Ronald Clark, *The life of Bertnand Russell*, Knopf, New York, p.176, 1976.

[2] Scott Elisha Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, Washington, DC, 1968

[3] G H Hardy, *Mathematician's Apology*, Cambridge University Press, New York, p.94, 1967.

Address for Correspondence
V G Tikekar
c/o Thakar
New Mahadwar Road
Near Padmaraje School
Kolhapur 416006, India.