Mayer–Jensen Shell Model and Magic Numbers

An Independent Nucleon Model with Spin-Orbit Coupling

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Many nuclear models have been put forward since 1932. Among them the collective model proposed by Aage Niels Bohr and Ben Roy Mottelson and the nuclear shell model proposed by Maria Goeppert Mayer and Johannes Hans D Jensen are the two most successful models. A number of experimental facts like the existence of magic numbers compiled by Maria Mayer led to the discovery of the nuclear shell model. The addition of a nuclear spin-orbit coupling force to the mean field of the nucleons successfully predicted the nuclear magic numbers and many other properties of nuclei.

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1. Introduction

One can legitimately claim that nuclear physics began with the discovery of radioactivity in 1896 by Becquerel though Rutherford and his colleagues discovered the nucleus only in 1911 from the results of their experiments on scattering of α-particles by a gold foil. Rutherford immediately proposed his model for the atom in which the positive charges are concentrated in a very small region, the dimension of which is nearly hundred thousand times smaller than that of the atomic radius and electrons orbiting around this core, well outside it, just like the planets that orbit around the sun.

According to classical electrodynamics, any accelerated charge must emit radiation. Since the orbiting electron is accelerated due to the Coulomb force between it and the positively charged nucleus, it should continuously emit electromagnetic radiation, thereby losing energy, and should therefore spiral in closer and closer to the

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nucleus until it comes into contact with it. The Rutherford atomic model was thus incompatible with classical physics. Further, it did not account for the experimentally observed details of the atomic spectra.

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The next major step to explain the atomic structure came in 1913 with Niels Bohr's quantum theory of the hydrogen spectrum based on Rutherford's model of the atom. Bohr's famous assumptions, which led him to a successful explanation of the spectrum of atomic hydrogen, are:

1. The orbital angular momentum of the electron is restricted to be an integral multiple of $h/2\pi$.
2. A quantum of energy is emitted when the electron changes from an orbit of higher energy to another of lower energy, the energy $h\nu$ carried by this quantum being equal to the difference in the energy of the two levels.
3. An electron in its orbit of lowest energy does not radiate.

Bohr tried to use these principles to explain the atomic structure of heavier elements, but with little success. Even for hydrogen, Bohr's theory could not explain the fine structure lines.

Further developments in our understanding of the atomic structure came through the de Broglie wave-particle duality. Following this idea of wave-particle duality, Heisenberg, Bohr, Schrödinger and Dirac established Quantum Mechanics which could explain these microscopic phenomena at the atomic level with theoretical methods which use ideas of probability. By 1928, a theory which could explain the atomic structure was well founded, but a theoretical description of nuclear structure was still in its infancy.

Quantum Mechanics can explain the atomic structure completely.

2. Nuclear Models

Even after the experimental discovery of the neutron by Chadwick in 1932, the experimental information available about the nuclei, discovered so famously by Rutherford, was rather sparse. The phenomenon of nuclear radioactivity, a not too accurate chart of a few known isotopes (nuclei with the same number of protons Z but differing number of neutrons N), an estimate of the nuclear radius (proportional to $A^{1/3}$, where A is the mass number) and a few nuclear reactions was all that was known. As time went on, experiments provided more accurate information on nuclear sizes, approximately given by $1.4 \times A^{1/3}$fm, (1fm $= 10^{-15}$m) and the nuclear spins as well as magnetic moments of the nuclei. Further, the nature of the strong nuclear force acting between the neutrons and protons (collectively called nucleons), inside the nucleus was not completely understood and a unique, closed form expression for this force such as is available for the Coulomb force between the electron and the nucleus in the atomic case, was not known. This made the problem of explaining the nuclear structure much more difficult than in the atomic case.

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In atoms, interaction of the electrons with each other affects their motion around the nucleus only in a small way. However, it is the strong interaction among the many nucleons in the nucleus that holds the nucleus together. Therefore, it is not possible to understand the structure of a nucleus in a simple manner, as is the case for atoms. There we just have to solve, using the tools of quantum mechanics, a simple two-body Coulomb force problem involving the electron and the nucleus in the atom. In fact, the nucleus is a unique, many body system. Neither does it have small enough number of particles, like the electrons and nucleus in an atom, with precisely known laws describing the force acting among them, hence amenable to analysis in terms of this two body force; nor does it have a huge number of parti-

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cles, like a mole of gas, where we know for sure that statistical methods can be applied to understand the macroscopic properties of the many-body system. As a result of these theoretical difficulties, at least in the early days, the motion of nucleons inside a nucleus and the nuclear structure, could only be modeled in an approximate manner. This was the genesis of the different nuclear models that were proposed historically.

Many nuclear models had been proposed since 1932. The liquid drop model was historically the first to successfully describe many of the observed nuclear properties. It is based on the analogy of a nucleus to a drop of incompressible fluid. In particular, it was able to explain why the nuclear binding energy is proportional to the nucleon number (A). Niels Bohr and John Wheeler used it to calculate successfully nuclear reaction cross-sections and to explain nuclear fission. The most successful of the nuclear models are:

- The spherical nuclear shell model.
- The nuclear collective model.

The former was proposed independently in 1949 by Maria Goeppert Mayer and by J Hans D Jensen and coworkers. This won them the Nobel Prize in physics in 1963. (For the life history of Maria Goeppert Mayer see the previous article in this issue of *Resonance*. A short biography of JHD Jensen is given in *Box* 1). The latter was developed in 1952 by Aage Bohr (son of Niels Bohr) and Ben Roy Mottelson following an earlier suggestion by James Rainwater and won them the Nobel Prize in physics in 1975. Interestingly, both Niels Bohr and his son Aage Bohr won Nobel Prizes.

In fact there are six father-son pairs (Thompson, Bragg, Bohr, Siegbahn, von Euler, Kornberg) winning the Nobel Prize, the Braggs winning it together, one father-daughter pair (the Curies) and one mother-daughter pair winning it together (the Curies).

3. Magic Numbers

The atomic shell structure is essential in explaining the chemical periodicity of the elements. In the periodic

Box 1. Biography of J.Hans D Jensen

J H D Jensen was born on 25 June 1907 in Hamburg, Germany. His father Karl Jensen was a gardener and mother was Helene Ohm Jensen. His outstanding performance in school won him a scholarship. He studied physics at the University of Hamburg. In 1932 he was awarded PhD degree by the University of Hamburg and obtained DSc from the same university. He was appointed lecturer in the university in 1937. In 1941 he moved to Hannover and joined as professor of theoretical physics at the Technische Hochschule. Jensen became professor at the University of Heidelberg in 1949. From 1955 he served as the co-editor of the journal *Zeitschrift für Physik*. Jensen, H E Suess and O Haxel, and Maria Goeppert Mayer proposed the shell model independently in 1949. Jensen told he got the idea of the spin-orbit coupling, as explanation for the magic numbers, one day while shaving!

Jensen first applied quantum theory to the study of crystals and investigated the properties of crystals under high pressure. He then became interested in building a theoretical model of the atomic nucleus. He later also showed that rays or particles discharged by the nuclei of radioactive atoms are caught within a crystal in a backward movement similar to the recoil of a rifle. Jensen died on 11 February 1973 in Heidelberg, Germany.

table, the appearance of an inert gas indicates that a particular electron shell or sub-shell is closed, to give this atom its special stable character. When the atomic number Z is equal to 2(He), 10(Ne), 18(Ar), 36(Kr), 54(Xe),... the chemical element is most stable and does not interact with other atoms to form molecules. In the atomic shell model, the filling of shells and sub-shells and the appearance of gaps in the energy spectrum of the shells (significant jumps in the electron energy in going from one shell to the next) are used to explain these magic numbers.

After 1932, experimental data on nuclei repeatedly revealed that in the nuclear case also, there exist a series of magic numbers, called so, because nuclei with these values for the atomic number Z and/or neutron number N, showed exceptional stability. It was found that when the proton number and/or the neutron number was equal to one of these *magic numbers*:

$$2,\ 8,\ 20,\ 28,\ (40),\ 50,\ 82,\ 126$$

then the nucleus is particularly stable and has a spherical shape. The evidence for the number 40 is weaker, that is why it is shown in parenthesis. These numbers are called magic for nuclei because of the stability of such nuclei. Although the magic numbers are different for those in atom, the presence of nuclear stability suggests some type of shell structure and shell closure as found in atoms. Magic numbers and many other experimental facts led to the development of nuclear shell model. We list these below in the next section.

A nucleus whose atomic number *Z* and/or neutron number *N* is equal to one of the magic numbers, shows exceptional stability.

4. Facts Supporting Nuclear Shell Model

The following experimental facts observed in the nuclei strongly suggest a shell structure for the nucleons in nucleus.

1. Beginning with H, Ca ($Z = 20$) is the first element with more than four stable isotopes including ${}^{40}_{20}$Ca and ${}^{48}_{20}$Ca indicating that Z and N of 20 and 28 have stability.
2. The element Sn ($Z = 50$) has 10 stable isotopes, which is more than any other element. The element ${}^{83}_{209}$Bi ($N = 126$) is the last stable isotope.
3. For $N = 20$, there are five stable isotones (same N but different Z): ${}^{16}_{36}$Sr, ${}^{17}_{37}$Cl, ${}^{18}_{38}$Ar, ${}^{19}_{39}$K and ${}^{20}_{40}$Ca.
4. In the even Z ($Z > 28$) nuclides, no isotope has an abundance of more than 60 percent, with three exceptions: ${}^{88}_{38}$Sr ($N = 50$), ${}^{138}_{56}$Ba ($N = 82$) and ${}^{140}_{28}$Ce ($N = 82$) indicating that the nuclei with neutron number 50 or 82 are more stable.
5. The separation energy for one proton or one neutron is very large for a nucleus with Z or N magic.
6. The probability of nuclei with neutron numbers 20, 28, 50, 82 or 126 capturing a neutron is much less than that of their neighbouring nuclei.

7. Unusually high first excited state energy is observed for the double magic nuclei $^{4}_{2}\mathrm{He}$, $^{16}_{8}\mathrm{O}$, $^{40}_{20}\mathrm{Ca}$, $^{48}_{20}\mathrm{Ca}$, $^{90}_{40}\mathrm{Zr}$ and $^{128}_{82}\mathrm{Pb}$. *Box* 2 gives the magic number nuclides.

5. Nuclear Shell Model

The existence of magic number may be explained by an independent-particle model. In this model, the nucleons do not interact with each other; the effect of all other nucleons in the nucleus is replaced by an average or mean field that binds individual nucleons to the nucleus. When one calculates in this picture, using quantum mechanics, the allowed energy levels that a single nucleon may occupy, one finds that there exist groups (shells) of

Box 2. Table of Magic Number Nuclides

Number of protons	2	8	20	28	50	82	126
	$^{4}\mathrm{He}$	$^{16}\mathrm{O}$	$^{40}\mathrm{Ca}$	$^{58}\mathrm{Ni}$	$^{112}\mathrm{Sn}$	$^{204}\mathrm{Pb}$	
		$^{17}\mathrm{O}$	$^{42}\mathrm{Ca}$	$^{60}\mathrm{Ni}$	$^{114}\mathrm{Sn}$	$^{206}\mathrm{Pb}$	
		$^{18}\mathrm{O}$	$^{43}\mathrm{Ca}$	$^{61}\mathrm{Ni}$	$^{115}\mathrm{Sn}$	$^{207}\mathrm{Pb}$	
			$^{44}\mathrm{Ca}$	$^{62}\mathrm{Ni}$	$^{116}\mathrm{Sn}$	$^{208}\mathrm{Pb}$	
			$^{46}\mathrm{Ca}$	$^{64}\mathrm{Ni}$	$^{117}\mathrm{Sn}$		
			$^{48}\mathrm{Ca}$		$^{118}\mathrm{Sn}$		
					$^{119}\mathrm{Sn}$		
					$^{120}\mathrm{Sn}$		
					$^{122}\mathrm{Sn}$		
					$^{124}\mathrm{Sn}$		
Number of neutrons	2	8	20	28	50	82	126
	$^{4}\mathrm{He}$	$^{15}\mathrm{N}$	$^{36}\mathrm{S}$	$^{48}\mathrm{Ca}$	$^{86}\mathrm{Kr}$	$^{136}\mathrm{Xe}$	$^{208}\mathrm{Pb}$
		$^{16}\mathrm{O}$	$^{37}\mathrm{Cl}$	$^{50}\mathrm{Ti}$	$^{87}\mathrm{Rb}$	$^{138}\mathrm{Ba}$	$^{209}\mathrm{Bi}$
			$^{38}\mathrm{Ar}$	$^{51}\mathrm{V}$	$^{88}\mathrm{Sr}$	$^{139}\mathrm{La}$	
			$^{39}\mathrm{K}$	$^{52}\mathrm{Cr}$	$^{89}\mathrm{Y}$	$^{140}\mathrm{Ce}$	
			$^{40}\mathrm{Ca}$	$^{54}\mathrm{Fe}$	$^{90}\mathrm{Zr}$	$^{141}\mathrm{Pr}$	
					$^{92}\mathrm{Mo}$	$^{142}\mathrm{Nd}$	
						$^{144}\mathrm{Sm}$	

closely spaced allowed levels with large gaps between these groups. The number of these closely spaced levels with large gaps, thus provides an understanding of the existence of the experimentally observed magic numbers.

The existence of magic numbers can be understood in terms of the existence of large energy gaps between groups of single particle states.

If the neutrons or protons fill completely the shell (closed shell) then the energy required to excite the nucleus, by transition of some of the nucleons to a higher energy level, is much larger than otherwise. In fact some knowledge about the force between two nucleons (or alternatively the inter-nucleon potential) was obtained by experimental study of the proton-neutron bound state (the deuteron) as well as from proton-proton and proton-deuteron scattering experiments. The experimental results can be equally well explained by differing functional dependence of this potential on the inter-nucleon distance and thus do not determine it uniquely. The average potential of the nuclear shell model can thus be taken to have any form from a simple square-well potential to the much more complicated deformed potential. Solving the Schrödinger equation will then give the allowed energy levels that a nucleon may occupy. Taking the average potential of the nucleon inside the nucleus to be the isotropic, three-dimensional simple harmonic potential, the number of nucleons needed to get a closed shell are then found to be

$$2,\ 8,\ 20,\ 40,\ 70,\ 112,\ 168,\ \ldots$$

Taking the average potential of the nucleon inside the nucleus to be the isotropic, three-dimensional simple harmonic potential, the number of nucleons needed to get a closed shell are then found to be 2, 8, 20, 40, 70, 112, 168, ...

Thus one sees that the harmonic oscillator potential accounts for the magic numbers 2, 8, 20 and 40. But a deviation is observed beyond these numbers. In fact, none of the potentials indicated by the two nucleon data was found to be capable of explaining all the magic numbers.

6. Spin-Orbit Energy

The departure of the experimentally observed sequence of magic numbers from those calculated in the simple picture above, in case of numbers beyond 40, was explained by taking into account the effect of single particle spin-orbit energy. This was suggested by Maria Goeppert Mayer as well as by Haxel, Jensen and Suess independently, in 1949. Their key step was the addition of a nuclear spin-orbit coupling force to the average field felt by each individual nucleon. They assumed the nucleus as a series of closed shells and that pairs of protons and neutrons prefer to couple together.

Maria Goeppert Mayer described her idea in the following (to her daughter)[1]: *Think of a room full of waltzers. Suppose they go round the room in circles, each circle closed within another. Each circle corresponds to an energy level. Then imagine that in each circle, you can fit twice as many dancers by having one pair goes clockwise and another pair goes counter-clockwise. Then add one more variation; all the dancers are spinning twirling round and round like tops as they circle the room, each pair both twirling and circling. But only some of those that go counter-clockwise are twirling counter-clockwise; the others are twirling clockwise while circling counter-clockwise. The same is true of those that are dancing around clockwise; some twirl clockwise, others twirl counter-clockwise.*

Nucleons of a given orbital angular momentum have two possible energies, depending on whether their spin is parallel or antiparallel to the orbital motion. This splitting of the energy levels is called *spin-orbit coupling*.

The point is that it is easier to spin in one direction than in the other direction. The couples spinning in the easier direction will need slightly lower energy than the couples spinning in the more difficult direction. Therefore, for a given circle of dancers, the energy necessary to orbit will be different for couples spinning in opposite senses. In the same way, nucleons of a given orbital angular momentum have two possible energies, depending on whether their spin is parallel or antiparallel to the orbital motion. This splitting of the energy levels is called *spin-orbit coupling*.

The nuclear spin-orbit coupling force is much stronger

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[1] S B McGrayne, *Nobel Prize Women in Science*, Joseph Henry Press, Washington DC, 1998.

and has opposite sign to the spin-orbit coupling force in atoms. The force which gives rise to the coupling of the spin and orbital motions of the nucleons in the nucleus has a different origin from the coupling of spin and orbit motion of electrons in atoms. The spin-orbit coupling in nuclei is related to the force between the nucleons and is not electromagnetic in origin as is the spin-orbit coupling in atoms.

The nuclear spin-orbit coupling does not arise due to electromagnetic interactions.

However, just as in atoms the nuclear spin-orbit coupling which is proportional to the scalar product of the orbital angular momentum operator $\vec{L}$ and the intrinsic spin operator $\vec{S}$, splits a nucleon energy level which is labelled by l into two levels $j = l \pm s$ (except for $l = 0$) but with a difference that the $j = l - 1/2$ lies higher in energy (less binding) and $j = l + 1/2$ lies lower in energy.

Box 3 gives order of energy levels obtained for a square-well average potential by including the spin-orbit coupling. It is the particular lowering of the $j = l + 1/2$ orbital of a given N oscillator shell, which is lowered into the orbits of the $(N - 1)$ shell, which accounts for the new shell closure numbers at 28 ($1f_{7/2}$ shell), 50 ($1f_{5/2}$ shell), 82 ($1g_{7/2}$ shell) and 126 ($1h_{9/2}$ shell).

The total number of nucleons in any level with quantum number j is $2j + 1$. Consider the number 28. The oscillator shell closes at 20. The next levels are 1f($l = 3$) and 2p($l = 1$). The 1f level splits into $j = 7/2$ and $j = 5/2$ with 7/2 lower. Because the energy difference is large, and the 7/2 level contains 8 states, we have the gap at $20 + 8 = 28$ nucleons. For the magic number 50, the 1g level splits into $j = 9/2$ and $j = 7/2$ with a large gap and $j = 9/2$ having lower energy. If all the levels upto $1g_{9/2}$ are filled, then we get the magic number 50. Similarly, when the levels upto $1k_{11/2}$ and $1i_{13/2}$ are filled we get the magic numbers 82 and 126 respectively. It is seen that due to the spin-orbit splitting, new energy levels and groups with gaps in their energies at magic

If all the levels upto $1g_{9/2}$ are filled, then we get the magic number 50, when the levels upto $1k_{11/2}$ and $1i_{13/2}$ are filled we get the magic numbers 82 and 126 respectively.

numbers appear. Thus the Mayer–Jensen shell model explains the magic numbers of nuclei.

7. Success and Failure of Shell Model

One of the major successes of the nuclear shell model was the prediction of the spins and parities of the ground states of most of the nuclei rather well. For nuclei with one nucleon outside of a closed shell, or one nucleon vacancy (hole) in a closed shell, the nuclear ground state spin and parity are determined by the extra nucleon or nucleon hole because the nucleons inside a closed shell have zero angular momentum just as in atoms. Another success of the nuclear shell model was the prediction of the islands of nuclear isomerism in regions around the closed shells. The shell model also correctly predicts the sign and roughly the magnitude of ground state magnetic moments of nuclei. The shell model predictions of the quadrupole moments are generally in agreement with experiments for nuclei near the magic numbers.

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However, the shell model predictions often differ drastically from experimental measurements for nuclei in other regions. Also, the shell model transition rates for γ-ray decay between nuclear energy levels are very different from the experimental values. Further, additional levels not predicted by shell model appear in many nuclei.

In the shell model of the nucleus, it is assumed that nucleons mainly move independently from each other in an average (mean) field with a large mean-free path. The single particle shell model based on the assumption of the existence of a spherically symmetric potential in the nucleus, plus a spin-orbit coupling term fails to explain some aspects of properties of a portion of nuclei such as their large quadrupole moments in many cases. Rainwater was the first scientist who tried to correct for this failure of the shell model by introducing the idea of de-

Box 3. Order of Energy Levels Obtained from the Square-well Potential by Spin-orbit Coupling

Oscillator number	Square-well	Spin term	Number of states	Shells	Total number
0	1s	$1s_{1/2}$	2	2	2
1	1p	$1p_{3/2}$	4	6	8
	$1p_{1/2}$	2			
2	1d	$1d_{5/2}$	6	12	20
		$1d_{3/2}$	4		
	2s	$2s_{1/2}$	2		
3	1f	$1f_{7/2}$	8	8	28
		$1f_{5/2}$	6	22	50
		$2p_{3/2}$	4		
	2p	$2p_{1/2}$	2		
		$1g_{9/2}$	10		
4	1g	$1g_{7/2}$	8	32	82
		$2d_{5/2}$	6		
	2d	$2d_{3/2}$	4		
	3s	$3s_{1/2}$	2		
		$1k_{11/2}$	12		
5	1k	$1h_{9/2}$	10	44	126
		$2f_{7/2}$	8		
	2f	$2f_{5/2}$	6		
		$3p_{3/2}$	4		
	3p	$3p_{1/2}$	2		
		$1i_{13/2}$	14		
6	1i	$1i_{11/2}$			
	2g				
	3d				
	4s				

formation in the shape of the nuclear core due to the motion of the loose odd nucleon outside the core in odd-A nuclei. Due to such interaction, the nucleus assumes a spheroidal shape, thus leading to high quadrupole moment. Aage Bohr and Mottelson futher elaborated the model by calculating the single particle energies in a deformed potential rather than the spherical potential used in the shell model.

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The shell model which had its origin in a study of properties of stable nuclei, while working very well in that case, fails in some exotic parts of the nuclear landscape where the shell gaps and magic numbers show deviations from those observed for the case of stable nuclei. One such region is the so-called *island of inversion* located around $Z = 10 - 12$ and $N = 2$ magic number, characterized by ground state deformations, which are surprisingly large for closed shell nuclei. Such behaviour can be explained only by a collapse of the usual filling of the single particle levels with neutrons occupying the pf shell before the lower sd shell is fully closed.

The regions near ^{32}Mg($N = 20$) contains several deformed nuclei. This deformation has been attributed to an inversion of the order in which the $f_{7/2}$ and $d_{3/2}$ orbitals are filled. Recently at the National Superconducting Laboratory's new coupled cyclotron facility in USA, ^{33}Al and ^{33}Mg have been produced by fragmentation of a 140 MeV/nucleon ^{40}Ar beam in a Be target. The priliminary studies on ^{33}Al, ^{33}Mg and also ^{31}Mg nuclei have proved that $Z = 20$ is not a magic number for these nuclei. This tells us that even today, there is theoretical interest and experimental activity related to the nuclear shell model.

Suggested Reading

[1] B K Mukherjee and D K Dhawan (Eds.), *Encyclopaedia of World Great Physicists*, Vol. 3, Nobel Lectures by M G Mayer, p.928 and Nobel Lectures by J H D Jensen, p.943, Overseas Press, New Delhi, 2006.

[2] K Heyde, *The Nuclear Shell Model*, Springer, Bristol, 1991.

[3] Nuclear Shell Model (http://www.halexandria.org/dward472.htm)

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