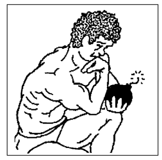
Think It Over



This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

The Shrinking Unit Ball

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The concepts of one, two and three dimensions are familiar to us. We think of a straightline as one dimensional, a plane as two dimensional and space as three dimensional. It is not easy to visualise objects in four or higher dimensional spaces. There are many results about high dimensional spaces that are somewhat counterintuitive. Here is one such result.

For an integer $n \geq 1$, let R^n be the n-dimensional Euclidean space. That is, R^n is the collection of all ordered n-tuples $\tilde{x} = (x_1, x_2, \ldots x_n)$ where each x_i is a real number. Then $S_n \equiv \{\tilde{x} : \tilde{x} \in R^n, \sum_1^n x_i^2 \leq 1\}$ is called the *unit ball* in n-space. It is the set of all points within a distance of one unit from the origin.

Thus S_1 is the line segment $[-1, +1]$ in R^1, S_2 is the unit disc $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ in R^2 and S_3 is the unit ball in R^3.

Let V_n be the 'volume' of S_n in R^n. For example, $V_1 = 2$, the length of S_1; $V_2 = \pi$, the area of S_2; $V_3 = \frac{4\pi}{3}$, the

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volume of S_3. Noting that

$$V_1 = \int_{S_1} 1 dx,$$

$$V_2 = \int\int_{S_2} 1 dx_1 dx_2,$$

$$V_3 = \int\int\int_{S_3} 1 dx_1 dx_2 dx_3,$$

i.e., the integral of the function $f(x) \equiv 1$ over S_1, S_2 and S_3 respectively, one can define V_n as

$$V_n \equiv \int\int\int_{S_n} 1 dx_1 dx_2 \ldots dx_n,$$

the Riemann integral of $f(x) \equiv 1$ over the set S_n in R^n. Seeing that $V_1 = 2 < V_2 = \pi < V_3 = \frac{4\pi}{3}$ one is tempted to conclude that V_n is increasing in n. It turns out that $V_n \to 0$ as $n \to \infty$. Can you show this?