

Relative Growth Rates

K B Athreya



K B Athreya is a Visiting Professor at IMI, Mathematics Department, IISc, Bangalore and is also a professor of mathematics and statistics at Iowa State University, Ames, Iowa, USA. His research interests include mathematical analysis, probability theory and its applications and statistics. His spare time is spent listening to Indian classical music.

If several objects grow with time the one with the highest rate of growth will dominate the rest over time. In this article this is discussed for three cases: when the number of objects is finite, countably infinite and a continuum.

If two objects, say A_1 and A_2 , grow at rates a_1 and a_2 per unit time respectively, then after n units of time, one unit each of A_1 and A_2 would become a_1^n and a_2^n respectively. Thus, if $1 < a_1 < a_2$ then both objects would grow to be very large as n gets large. On the other hand, the relative growth of A_2 with respect to (w.r.t.) A_1 will be $(a_2/a_1)^n$ and that will go to infinity while that of A_1 w.r.t. A_2 will be $(a_1/a_2)^n$ and that will go to zero.

For example, if there are two fixed deposit schemes, say, S_1 and S_2 , which are compounded annually with interest rates r_1 and r_2 respectively, then one unit of money invested in each of S_1 and S_2 would grow in n years to $(1 + r_1)^n$ and $(1 + r_2)^n$ respectively.

If $r_1 < r_2$ and $\frac{1+r_2}{1+r_1} = 1 + \theta$ for some $\theta > 0$ then, the relative growth of S_2 w.r.t. S_1 will be $(1 + \theta)^n$ which is at least $1 + n\theta$. Thus, in $n > \frac{1}{\theta}$ years, one unit in scheme S_2 grows to more than twice to what one unit in S_1 grows to.

More generally, let there be k objects, A_1, A_2, \dots, A_k that grow at rates a_1, a_2, \dots, a_k respectively. Suppose for some r , $1 \leq r \leq k$, $a_r > a_i$ for all $i \neq r$, i.e. a_r is the unique maximum among a_1, a_2, \dots, a_k . Then one unit each of A_1, A_2, \dots, A_k in n units of time grows into $a_1^n, a_2^n, \dots, a_k^n$.

Keywords

Relative proportion, Gaussian cumulative distribution function, standard normal distribution, Riemann integrable function.



Let

$$p_{ni} \equiv \frac{a_i^n}{\sum_{j=1}^k a_j^n}, \quad 1 \leq i \leq k \quad (1)$$

denote the relative proportion of the size of A_i to the sum of the sizes of all A_j 's at time n . Then, it is easy to conclude that

$$p_{nr} \rightarrow 1 \quad \text{and} \quad \sum_{\substack{i \neq r \\ i=1}}^k p_{ni} \rightarrow 0 \quad (2)$$

as $n \rightarrow \infty$. Thus, object A_r tends to dominate all the rest over time. For example, if there are k countries in the world with annual population growth rates a_1, a_2, \dots, a_k and if there is one country whose growth rate is higher than others, then after some time that country will have the largest population and the proportion of people in that country to the total population of the world will go to one.

What about countably infinite number of objects? Let A_1, A_2, A_3, \dots , be a countably infinite collection of objects with growth rates a_1, a_2, \dots per unit time respectively. Assume that

(i) for some r , $a_r > a_i, i \neq r$

(ii)

$$\sum_{j=1}^{\infty} \left(\frac{a_j}{a_r} \right)^{n_0} < \infty \text{ for some } n_0 \geq 1 \quad (3)$$

Since one unit of object A_i would become a_i^n after n units of time, the relative proportion of A_i to the total of all objects after n units of time, p_{ni} say will be

$$p_{ni} \equiv \frac{a_i^n}{\sum_{j=1}^{\infty} a_j^n}$$



using condition (ii) and the fact $0 < \frac{a_j}{a_r} < 1$ for all $j \neq r$ it can be shown that

$$\lim_{n \rightarrow \infty} \sum_{j \neq r} \left(\frac{a_j}{a_r} \right)^n = 0. \tag{4}$$

Thus,

$$p_{nr} \rightarrow 1 \tag{5}$$

and

$$\sum_{i \neq r} p_{ni} \rightarrow 0 \tag{6}$$

as $n \rightarrow \infty$.

Next, what about a continuum of objects?

One could try something like this. Let $f(x)$ be a non-negative function on some interval $[a, b]$. Associate with each point x in $[a, b]$ an object A_x and think of $f(x)$ as the growth rate of A_x . Thus one unit of A_x will grow into $(f(x))^n$ in n units of time. Call $f(x)$ the rate function. Let it have a unique maximum occurring at some x_0 , $a \leq x_0 \leq b$. That is, $f(x_0) > f(x)$ for all $x \neq x_0$. Then, in analogy with (2) and (5), it is not unreasonable to ask whether

$$\frac{(f(x_0))^n}{\int_a^b (f(x))^n dx} \rightarrow 1 \tag{7}$$

and

$$\frac{(f(x'))^n}{\int_a^b (f(x))^n dx} \rightarrow 0 \quad \text{for all } x' \neq x_0. \tag{8}$$

Unfortunately, this does not hold fully.

For example, if $f(x) \equiv x$ and $[a, b] = [0, 1]$

then $x_0 = 1$ and $\int_0^1 (f(x))^n dx = \frac{1}{(n+1)}$ so that

$$\frac{(f(x_0))^n}{\int_0^1 (f(x))^n dx} = (n+1) \rightarrow \infty$$



and hence (7) does not hold.

However, for $0 \leq x' < x_0 = 1$

$$\frac{(f(x'))^n}{\int_0^1 (f(x))^n dx} = (n+1)(x')^n$$

does go to zero as $n \rightarrow \infty$, i.e. (8) holds. This suggests that an appropriate generalisation of (2) and (5) might be as follows. For every $\varepsilon > 0$,

$$p_n(x_0, \varepsilon) \equiv \frac{\int_{x_0-\varepsilon}^{x_0+\varepsilon} (f(x))^n dx}{\int_a^b (f(x))^n dx} \rightarrow 1 \quad (9)$$

as $n \rightarrow \infty$ or equivalently

$$1 - p_n(x_0, \varepsilon) = \frac{\int_{|x-x_0| \geq \varepsilon} (f(x))^n dx}{\int_a^b (f(x))^n dx} \rightarrow 0 \quad (10)$$

as $n \rightarrow \infty$. That is at time n , the proportion of objects that lie in any nondegenerate two sided neighborhood of x_0 to the total number of objects (rather than those that lie exactly at x_0) goes to one and the proportion outside any such neighborhood goes to zero.

It turns out that (9) and (10) hold under some reasonable conditions on $f(x)$. Assume that the rate function $f(\cdot)$ is such that

- (i) $f = [a, b] \rightarrow R^+ = [0, \infty)$,
- (ii) for some x_0 in $[a, b]$, $f(x_0) > f(x)$ for all $x \neq x_0$, x in $[a, b]$
- (iii) $f(\cdot)$ is continuous at x_0
- (iv) for each $\varepsilon > 0$, there is a λ_ε such that $0 \leq \lambda_\varepsilon < 1$ and $|x - x_0| \geq \varepsilon$ implies $0 \leq f(x) \leq \lambda_\varepsilon f(x_0)$.

(11)



Then, we claim that for each $\varepsilon > 0$, (9) and (10) hold.

Here is a proof of this claim. Fix $\varepsilon > 0$. Let λ_ε be as in condition (iv) above. By condition (iii) above, i.e. the continuity of $f(\cdot)$, and the fact $f(x_0) > 0$, for any $0 \leq \lambda < 1$, we can choose an $\eta > 0$ such that $|x - x_0| < \eta$ implies $f(x) > \lambda f(x_0)$. Now choose the λ to be less than λ_ε . Then the denominator of (10) is

$$\geq \int_{|x-x_0|<\eta} (f(x))^n dx \geq \lambda^n (f(x_0))^n \eta$$

and by condition (iv) the numerator of (10) is

$$\leq (\lambda_\varepsilon f(x_0))^n (b - a)$$

Thus

$$0 \leq 1 - p_n(x_0, \varepsilon) \leq \left(\frac{\lambda}{\lambda_\varepsilon}\right)^n \frac{(b - a)}{\eta}$$

Since $0 \leq \frac{\lambda}{\lambda_\varepsilon} < 1$, the rightside above goes to zero as $n \rightarrow \infty$, establishing (10) which implies (9). Note further that for each $\varepsilon > 0$

$$\begin{aligned} 0 &\leq \frac{\int_a^b (f(x))^n dx}{(f(x_0))^n} \leq \int_{|x-x_0|<\varepsilon} \left(\frac{f(x)}{f(x_0)}\right)^n dx \\ &\quad + \int_{\varepsilon \leq |x-x_0|} \left(\frac{f(x)}{f(x_0)}\right)^n dx \\ &\leq 2\varepsilon + \lambda_\varepsilon^n (b - a) \end{aligned}$$

and hence that

$$0 \leq \lim_{n \rightarrow \infty} \frac{\int_a^b (f(x))^n dx}{(f(x_0))^n} \leq 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary this says that

$$\lim_{n \rightarrow \infty} \frac{\int_a^b (f(x))^n dx}{(f(x_0))^n} = 0$$

Thus in (9) both the numerator and denominator when divided by $(f(x_0))^n$ go to zero as $n \rightarrow \infty$ and yet their ratio goes to one.



One can refine (9) if we know more about the behaviour of $f(x)$ near the optimum value x_0 . The assertion in (9) says that for each $\varepsilon > 0$ the proportion of objects in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ goes to one.

One could ask whether we could find smaller intervals O_n around x_0 that get small and shrink to x_0 as $n \rightarrow \infty$ such that the proportion in n still goes to one. For example, would $O_n = (x_0 - \varepsilon_n, x_0 + \varepsilon_n)$ where $n\varepsilon_n \rightarrow 0$ do? What if $\sqrt{n\varepsilon_n} \rightarrow 0$.

The following special case was considered by S Shirali in an earlier issue of *Resonance* [1]. Let $f(x) = (\cos x)^n$ be the rate function on the interval $[a, b] = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then f satisfies (11) and has a unique maximum of 1 at $x_0 = 0$. Thus

$$\int_{-\pi/2}^{\pi/2} (\cos x)^n dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

Shirali [1] asserts that for any $-\infty < \alpha < \beta < \infty$

$$\frac{\int_{\frac{\alpha}{\sqrt{n}}}^{\frac{\beta}{\sqrt{n}}} (\cos x)^n dx}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^n dx} \rightarrow \Phi(\beta) - \Phi(\alpha) \quad (12)$$

as $n \rightarrow \infty$, where

$$\Phi(x) \equiv \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \quad (13)$$

is the standard normal (gaussian) cumulative distribution function, well known in probability theory and statistics and in many areas of science and engineering.

Thus, the proportion of objects in the neighborhood $(\frac{\alpha}{\sqrt{n}}, \frac{\beta}{\sqrt{n}})$ after n units of time converges as $n \rightarrow \infty$ to a quantity that is strictly positive but also strictly less than one. This implies that most of the objects are centered around the origin at a distance of the order \sqrt{n} . This also implies that the proportion of objects in



$O_n \equiv (-\varepsilon_n, \varepsilon_n)$ goes to zero if $\sqrt{n}\varepsilon_n \rightarrow 0$ (i.e. $\varepsilon_n \rightarrow 0$ faster than $\frac{1}{\sqrt{n}}$) and to one if $\sqrt{n}\varepsilon_n \rightarrow \infty$.

Another way of interpreting Shirali's assertion (12) is that if at time n one chooses an object at random with uniform distribution among all the objects and its location is X_n then the probability that X_n lies between $(\frac{\alpha}{\sqrt{n}}, \frac{\beta}{\sqrt{n}})$ converges to $\Phi(\beta) - \Phi(\alpha)$ or that the probability that $\sqrt{n}X_n$ lies in (α, β) converges to $\Phi(\beta) - \Phi(\alpha)$. In the language of probability theory it says that $\sqrt{n}X_n$ converges in distribution to a standard normal distribution.

Now let us prove (12). Note that

$$\sqrt{n} \int_{\frac{\alpha}{\sqrt{n}}}^{\frac{\beta}{\sqrt{n}}} (\cos x)^n dx = \int_{\alpha}^{\beta} \left(\cos \frac{u}{\sqrt{n}} \right)^n du$$

Now write

$$\begin{aligned} \cos \frac{u}{\sqrt{n}} &= \left(1 + \frac{\left(\cos \frac{u}{\sqrt{n}} - 1 \right) n}{n} \right) \\ &= \left(1 + \frac{\lambda_n(u)}{n} \right) \text{ say} \end{aligned} \quad (14)$$

Now for $u \neq 0$

$$\lambda_n(u) = \frac{\left(\cos \frac{u}{\sqrt{n}} - 1 \right) u^2}{\frac{u^2}{2n}}.$$

Since $\frac{\cos \theta - 1}{\theta^2} \rightarrow -1/2$ as $\theta \rightarrow 0$, $\lambda_n(u) \rightarrow -\frac{u^2}{2}$ as $n \rightarrow \infty$.

Next,

$$\left(1 + \frac{x_n}{n} \right)^n \rightarrow e^x \text{ if } x_n \rightarrow x.$$

So

$$\varphi_n(u) \equiv \left(\cos \frac{u}{\sqrt{n}} \right)^n \rightarrow e^{-u^2/2} \equiv \varphi(u),$$



say as $n \rightarrow \infty$.

It can be shown that $\varphi_n(\cdot)$ converges uniformly to $\varphi(u)$ on any finite interval $[\alpha, \beta]$, i.e. $\max\{|\varphi_n(u) - \varphi(u)| : \alpha \leq u \leq \beta\} \rightarrow 0$ as $n \rightarrow \infty$.

This implies that

$$\int_{\alpha}^{\beta} \left(\cos \frac{u}{\sqrt{n}} \right)^n \rightarrow \int_{\alpha}^{\beta} e^{-u^2/2} du$$

i.e.

$$\sqrt{n} \int_{\frac{\alpha}{\sqrt{n}}}^{\frac{\beta}{\sqrt{n}}} (\cos x)^n dx \rightarrow \int_{\alpha}^{\beta} e^{-u^2/2} du. \quad (15)$$

A little more effort can be used to show that

$$\sqrt{n} \int_{-n}^n (\cos x)^n dx \rightarrow \int_{-\infty}^{\infty} e^{-u^2/2} du. \quad (16)$$

Now

$$\int_{-\infty}^{\infty} e^{-u^2/2} du = 2 \int_0^{\infty} e^{-u^2/2} du = 2I, \text{ say,}$$

then

$$\begin{aligned} I^2 &= \left(\int_0^{\infty} e^{-u^2/2} du \right) \left(\int_0^{\infty} e^{-v^2/2} dv \right) \\ &= \int_0^{\infty} \int_0^{\infty} e^{-\frac{u^2+v^2}{2}} dudv \\ &= \int_0^{\infty} \int_0^{\pi/2} e^{-r^2/2} r dr d\theta \\ &= \left(\int_0^{\infty} e^{-r^2/2} r dr \right) \left(\int_0^{\pi/2} d\theta \right) \\ &= \left(\int_0^{\infty} e^{-t} dt \right) \frac{\pi}{2} = \frac{\pi}{2} \end{aligned}$$

Thus

$$2I = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}. \quad (17)$$

Following the above arguments it is not difficult to establish the following:

Let $f(x)$ be a Riemann integrable function on $[-1, 1]$ such that (i) $f(0) = 1$, (ii) $0 \leq f(x) \leq 1$ for all $|x| \leq 1$, (iii) f is continuous at 0 and (iv) for any $\varepsilon > 0$, there is a $0 < \lambda_\varepsilon < 1$ such that for all $|x| \geq \varepsilon$, $|f(x)| \leq \lambda_\varepsilon$. Then

(a) $A_n = \int_{-1}^1 (f(x))^n dx \rightarrow 0$ as $n \rightarrow \infty$.

(b) For each $\varepsilon > 0$,

$$p_n(\varepsilon) \equiv \frac{\int_{-\varepsilon}^{\varepsilon} (f(x))^n dx}{A_n} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

(c) If in addition to condition (i) and (iv) above if f satisfies

(v) For some $\alpha > 0$

$$\lim_{n \rightarrow 0} \frac{1 - f(x)}{|x|^\alpha} \equiv c_\alpha, \quad 0 < c_\alpha < \infty,$$

then

$$\begin{aligned} B_n(a, b) &= \frac{\int_{a/n^{1/\alpha}}^{b/n^{1/\alpha}} (f(x))^n dx}{A_n} \\ &\rightarrow \frac{\int_a^b e^{-c_\alpha |u|^\alpha} du}{\int_{-\infty}^{\infty} e^{-c_\alpha |u|^\alpha} du} \end{aligned} \quad (18)$$

Note that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-c_\alpha |u|^\alpha} du &= 2 \int_0^{\infty} e^{-c_\alpha v} \frac{1}{\alpha} v^{\frac{1}{\alpha}-1} dv \\ &= \frac{2}{\alpha c_\alpha^{1/\alpha}} \int_0^{\infty} e^{-u} u^{\frac{1}{\alpha}-1} du \\ &= \frac{2}{\alpha} \cdot \frac{1}{c_\alpha^{1/\alpha}} \cdot \Gamma(1/\alpha) \end{aligned}$$



where $\Gamma(p)$, the *gamma function*, is defined for $p > 0$ as

$$\Gamma(p) = \int_0^{\infty} e^{-u} u^{p-1} du.$$

It can be shown that $\Gamma(1) = 1$,

$$\Gamma(p + 1) = p\Gamma(p) \quad \text{for all } p > 0, \quad \Gamma(1/2) = \sqrt{\pi}.$$

Thus, if the rate function satisfies (i) to (v) above and one chooses an organism at random from the big collection then its location, X_n will be such that it is of order $n^{-1/\alpha}$ and $n^{1/\alpha} X_n$ will converge in distribution to a probability distribution on $(-\infty, \infty)$ with probability density function proportional to $e^{-c_\alpha |u|^\alpha}$.

Another formulation of (5) is the following:

Let $h(\cdot)$ be a Riemann integrable function on some bounded interval $[a, b]$ such that it achieves a unique maximum at x_0 . Let for $\varepsilon > 0, t > 0$

$$p(x_0, \varepsilon, t) \equiv \frac{\int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{th(x)} dx}{\int_a^b e^{th(x)} dx}$$

Then for each $\varepsilon > 0$,

$$p(x_0, \varepsilon, t) \rightarrow 1 \text{ as } t \rightarrow \infty.$$

Thus a probability distribution on $[a, b]$ with probability density proportional to $e^{th(x)}$ converges in distribution to the unit mass at x_0 where $h(\cdot)$ achieves its unique maximum. This is known in physics as the Laplace's method of locating the maximum of h and is related to what is called simulated annealing in optimization problems in operations research.

Suggested Reading

- [1] S A Shirali, *Solution to On a Use of Normal Distribution, Resonance, Vol.11, pp.88–90, 2006.*

Address for Correspondence

K B Athreya
 IMI
 Mathematics Department
 Indian Institute of Science
 Bangalore 560 012, India.
 Email:
 kba@math.iisc.ernet.in
 kba@instate.edu

