Odd Behaviour of the Even Integer 2

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It is always of interest when one finds a property that is true only of some given object, or some given class of objects. For example, if $n > 1$ is an integer, then $(n-1)!+1$ is divisible by n just when n is a prime number. In this article we look at some properties that are true only for the integer 2.

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It happens once in a while that we find a non-trivial property possessed by just one positive integer, or by some very small set of positive integers. When we do find such a property, it is an occasion to celebrate! A highly non-trivial (and much celebrated) example is the result of Wiles-Fermat: *The only integer $n > 1$ for which the equation $x^n + y^n = z^n$ has solutions in positive integers is $n = 2$.* Here are some other examples of properties possessed by just one or by very few integers:

1. *Every positive integer can be written as a sum of distinct powers of* 2. *If we replace '2' by any larger integer, then this claim becomes false.* (Here we include $1 = k^0$ in the set of powers of any positive integer k.) For example, 5 and 6 cannot be written as sums of distinct powers of 3.

2. *In the set of positive integers, the only square which differs by* 1 *from a cube is* 9. (Here we have $9-8 = 1$ or $3^2 - 2^3 = 1$.)

3. Consider the set A of composite integers $n > 1$ with the property that for every integer k such that $1 < k < n$, if k and n are coprime, then k is prime. Example: $6 \in A$. The following is true: A *has just eight elements, and its largest element is* 30.

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Arithmetic progression, odd and even, partial sum, square number, Pythagorean triple.

4. The following identity, often given as an exercise in induction, is well known: for all positive integers n,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

   This tells us that for $k = 3$, the sum $1^k + 2^k + 3^k + \cdots + n^k$ is a square for every positive integer n.

   Is there any other exponent k for which this is true? The answer is: *No; 3 is the only such exponent.*

5. *The only integer exceeding 1 which is both a Fibonacci number and a square is 144.* (The Fibonacci sequence goes: 0, 1, 1, 2, 3, 5, 8, 13, ... ; its defining property is that each term after the second one is the sum of the preceding two terms, e.g., $13 = 5 + 8$. It has been featured in earlier **Resonance** articles.)

6. *The only integer exceeding 1 which is both a Fibonacci number and a cube is 8.*

Of course, numerous such cases can be catalogued, and we shall not extend the list further. In passing, we note that item #1 on this list is well known and easy to prove, while items #2 and #3 are less well known and offer pleasant challenges. But items #4, #5 and #6 are very challenging indeed.

Remark

The phrase "non-trivial" (used at the start of this article) is hard to pin down in any precise way, but clearly we will want to exclude statements with single-line justifications, e.g., statements like:

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- *The only prime number which is even is 2.*

- *The only prime number which is both a Mersenne prime and a Fermat prime is* 3*.* (A Mersenne number is 1 *less* than a power of 2; a Fermat number is 1 *more* than a power of 2.) Also: *The only prime number which is* 1 *less than a square is* 3*.*

- *The only prime number which belongs to two different sets of twin primes is* 5*.* (Twin primes are pairs of primes differing by 2.) Also: *The only prime number which is* 4 *more than a fourth power is* 5*.* (Here we have $5 = 4 + 1^4$.)

- *The smallest prime number that cannot be written as a sum of three or fewer squares is* 7*.* (We know, after Lagrange, that every prime number can be written as a sum of four or fewer squares, and also that not every prime number can be written as a sum of three or fewer squares.) Also: *The only prime number which is* 1 *less than a cube is* 7*.*

- *The smallest prime number* p *that contradicts the assertion*

  "If p is a prime number, then so is $2^p - 1$"

  is $p = 11$*.* (Indeed, $2^{11} - 1 = 23 \times 89$.) Also: *The smallest repunit prime number is* 11*.* (A repunit prime is one all of whose digits are 1's, when written in base ten.)

The Number 2

In this article we feature the number **2**, which is the unique possessor of several properties; some are of a very deep nature (e.g., the Wiles-Fermat result). We shall prove three elementary properties associated with 2 *and with no other positive integer.*

The number **2** is the unique possessor of several properties.

The first two properties relate to arithmetical progressions with common difference 2; the first one concerns the *partial sums of the odd numbers*, while the second one concerns *schemes for generating Pythagorean triples* using the sequences of odd numbers and of even numbers, respectively. The third property concerns the *sets of integers that lie between successive k-th powers*, for $k = 2, 3, \ldots$, and a special property that holds only for the case $k = 2$.

1. *Partial Sums of the Odd Numbers*

The following identity for the partial sums of the sequence of odd numbers,

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2,$$

is well known; it even has an attractive pictorial proof (see *Box* 1).

Box 1. Pictorial Proof of an Algebraic identity

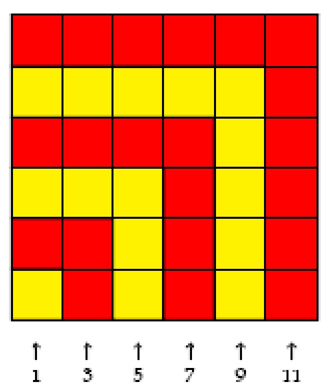
The aim is to show *pictorially* that

$$1 + 3 + 5 + \cdots + (2n - 1) \; = \; n^2,$$

for all positive integers n. Here is one way of doing so.

In the diagram, the strips alternately coloured yellow and red have 1, 3, 5, 7, 9, 11, ... squares, respectively, and it is only natural that

$$\begin{aligned} 1 &= 1^2, \\ 1 + 3 &= 2^2, \\ 1 + 3 + 5 &= 3^2, \\ 1 + 3 + 5 + 7 &= 4^2, \\ 1 + 3 + 5 + 7 + 9 &= 5^2, \end{aligned}$$



and so on. And that's the proof!

Noting that the sequence of odd numbers 1, 3, 5, 7, ... is an arithmetical progression (with common difference 2), we ask:

> *Is there any other arithmetical progression with the property that all its partial sums are squares?*

To avoid triviality, we want to exclude answers which are just the above progression multiplied by a square number

To avoid triviality, we want to exclude answers which are just the above progression multiplied by a square number (e.g., 4, 12, 20, 28, ...). We shall show that with this proviso, the answer is ***No***; there is no other such progression.

For the proof, we shall suppose that a and d are integers such that $d > 0$ and the arithmetical progression a, $a+d$, $a+2d$, $a+3d$, ... has the property described above. Then the sum

$$a + \big(a+d\big) + \big(a+2d\big) + \big(a+3d\big) + \cdots + \big(a+(n-1)d\big) = \frac{n\big(2a+(n-1)d\big)}{2}$$

is, by hypothesis, a square for every positive integer n.

Let $n = p$, an odd prime number. If the quantity $\frac{1}{2}p(2a+(p-1)d)$ is to be a square, the quantity $2a+(p-1)d = 2a+pd-d$ must be a multiple of p. This implies that $2a-d$ itself must be a multiple of p. Since p is an arbitrary odd prime number in this statement, it follows that $2a-d$ must be divisible by every odd prime number, and therefore that $2a-d=0$, i.e., $d=2a$. So the progression must be a, $3a$, $5a$, $7a$, ... , and for the stated property to hold, a itself must be a square. As we want to exclude trivial cases, we take $a = 1$, and this yields the sequence of odd numbers, as promised. □

2. *Generating Pythagorean Triples*

A *Pythagorean triple* is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. Examples: $(3, 4, 5)$ and

$(5, 12, 13)$. If no integer exceeding 1 divides all the three integers, then the triple is said to be *primitive*; we call it a PPT for short. Multiplying the integers of such a triple by any integer exceeding 1 yields a Pythagorean triple which is not primitive, e.g., $(6, 8, 10)$; but such triples are not of much interest.

The following scheme for generating PPTs is sometimes empirically hit upon by students. *Take a pair of consecutive odd numbers. Find the sum of their reciprocals and write it in its lowest terms as* a/b. *Then the triple* $(a, b, b+2)$ *is a PPT*. Example: Using the pair $\{3, 5\}$ we get the sum $8/15$, and from it the PPT $(8, 15, 17)$.

Following the discovery, we soon find that the even numbers do not have to be left out of the fun. Indeed, we have: *Take a pair of consecutive even numbers. Find the sum of their reciprocals and write it in its lowest terms as* a/b. *Then the triple* $(a, b, b+1)$ *is a PPT*. Example: Using the pair $\{4, 6\}$ we get the sum $5/12$, and from it the PPT $(5, 12, 13)$.

Proving that these schemes "work" is an easy exercise in school algebra. (Note, however, that not all PPTs can be generated by these schemes.)

The occurrence of the odd numbers and the even numbers in the above two schemes naturally prompts the following question:

> *Is there any other arithmetical progression with the property that by adding the reciprocals of arbitrary pairs of consecutive terms we are able to generate PPTs?*

We shall show that here too, the answer is ***No***; there is no other such progression.

For the proof, we shall suppose that a and d are integers such that $d > 0$ and the arithmetical progression a, $a+d$,

$a+2d$, $a+3d$, ... has the property that for any positive integer n, when we compute the sum of the reciprocals of the n-th and $(n+1)$-th terms, namely,

$$\frac{1}{a+(n-1)d}+\frac{1}{a+nd} = \frac{2a+(2n-1)d}{\big(a+(n-1)d\big)\cdot\big(a+nd\big)},$$

and write it as u/v, the quantity u^2+v^2 is a square. Note that for the analysis we do not have to worry about reducing the fraction to lowest terms, because the presence of un-canceled factors does not disturb the property in question. So we need to find pairs of integers a and $d>0$ such that the quantity

$$F(a,d,n) := \Big(2a+(2n-1)d\Big)^2 + \Big(\big(a+(n-1)d\big)\cdot\big(a+nd\big)\Big)^2$$

is a square for every positive integer n.

To check whether we are on the right track, let us see what happens when $a=1$, $d=2$ (this choice generates the sequence of odd numbers); we get, on simplifying:

$$F(1,2,n) = 16n^4+8n^2+1 = \big(4n^2+1\big)^2;$$

so $F(1,2,n)$ is always a square. For $a=2$, $d=2$ (this choice generates the sequence of even numbers), we get, on simplifying:

$$F(2,2,n) = 4\cdot\Big(4n^4+8n^3+8n^2+4n+1\Big) = \Big(2\left(2n^2+2n+1\right)\Big)^2;$$

so $F(2,2,n)$ too is always a square. Now we ask: *Will this happen for any other choice of a and d?*

Simplifying the form of $F(a,d,n)$ we get a fourth-degree polynomial:

$$\begin{aligned} &F(a,d,n) = \\ &n^4d^4+2n^3d^3(2a-d)+n^2d^2\left(6a^2-6ad+d^2+4\right) \\ &\quad +2nd\left(2a^3-3a^2d+a(d^2+4)-2d\right) \\ &\quad +a^4-2a^3d+a^2(d^2+4)-4ad+d^2. \end{aligned}$$

Extracting the square root of this polynomial in the old fashioned way, we find that the leading term (of degree 2) is n^2d^2; the next term (of degree 1) is $nd\,(2a-d)$; and the next term (of degree 0) is $2+a^2-ad$. But there is a remainder left at the end, and we find that

$$F(a,d,n) - \Big(n^2d^2 + nd(2a-d) + (2+a^2-ad)\Big)^2 = d^2-4,$$

a happy simplification!

This tells us that $F(a,d,n)$ is equal to a square plus d^2-4, and so will not be a square for large enough n unless $d^2-4=0$, i.e., $d=2$. This corresponds to an arithmetical progression with common difference 2; and it yields, as promised, the sequences of odd numbers (for odd a) and even numbers (for even a). □

3. Squares and Cubes

This section is based on Problem 11121 from the August-September 2006 issue of the American Mathematical Monthly.

Let n, k be positive integers. We ask, "*Can one find k distinct integers strictly between n^k and $(n+1)^k$, whose product is a perfect k^{th} power?*" For example, if $k=3$ and $n=2$, the answer is "Yes"; we may take the three integers to be $9,12,16$: they lie between $2^3=8$ and $3^3=27$, and their product is 12^3. The answer is again "Yes" if $k=3$ and $n=3$, as may be empirically checked. The answer for the general case turns out to be the following:

A. For $k=2$, the answer is "No" for all $n\geq 1$.

B. For $k=3$, the answer is "No" for $n=1$, but "Yes" for all $n>1$.

C. For $k>3$, the answer is "Yes" for all sufficiently large values of n.

We shall give the proofs of A and B here.

Proof of A. Suppose, to the contrary, that for some $n \geq 1$ there exist integers a, b strictly between n^2 and $(n+1)^2$, $a < b$, such that ab is a square. Write $a = uv^2$ and $b = UV^2$ where u and U are square-free. As ab is a square, so is uU; this implies that $u = U$. So we have $a = uv^2$ and $b = uV^2$ for some positive integers u, v, V with $v < V$. We must have $u > 1$, else we would have $n^2 < v^2 < V^2 < (n+1)^2$, which is absurd. We now get:

$$\frac{n}{\sqrt{u}} < v < V < \frac{n+1}{\sqrt{u}},$$

but this too is absurd, as $\frac{n+1}{\sqrt{u}} - \frac{n}{\sqrt{u}} = \frac{1}{\sqrt{u}} < 1$, whereas $V - v \geq 1$. So such a situation cannot happen. That is, integers a, b with the stated properties do not exist. □

Proof of B. We must find distinct integers a, b, c strictly between n^3 and $(n+1)^3$ such that abc is a cube. For $n = 1$ this is not possible; this may be checked by trial and error, as the set of available integers is $2, 3, 4, 5, 6, 7$. For $n > 1$ we shall give a simple rule for getting the required a, b, c: let a be the smallest square strictly larger than n^3, say $a = u^2$; then let $b = u(u+1)$ and $c = (u+1)^2$. This choice certainly makes abc a cube, for it yields $abc = u^3(u+1)^3$. But now we must show that a, b, c lie between n^3 and $(n+1)^3$. It suffices to show that $c < (n+1)^3$, i.e., $(u+1)^2 < (n+1)^3$. We verify empirically that this is the case when $n = 2$, for we get $u^2 = 9$ and $(u+1)^2 = 16 < 27$. Now suppose that the inequality fails for some $n > 2$; then we have

$$(u-1)^2 \leq n^3 < (n+1)^3 \leq (u+1)^2.$$

This yields $(u+1)^2 - (u-1)^2 > (n+1)^3 - n^3$, i.e., $4u > 3n^2 + 3n + 1$. We also have $u - 1 \leq n^{3/2}$. These two inequalities yield

$$3n^2 + 3n + 1 < 4n^{3/2} + 4.$$

Since $n > 2$ this implies that $3n^2 + 3n < 5n^{3/2}$, or $n^{1/2} + n^{-1/2} < \frac{5}{3}$. But this inequality is false for all $n > 0$; indeed, $n^{1/2} + n^{-1/2}$ cannot be less than 2.

So it must be that $(u+1)^2 < (n+1)^3$, and a, b, c do lie in the required interval. This proves the claim. $\square$

For higher values of k a similar argument may be used to prove the stated result.

Closing Remark

One may continue in this vein and catalogue more and more properties that are uniquely held by the number 2, adding to the list its occurrence in numerous natural laws. In view of its also being the only even prime, we feel justified in calling it *the oddest number*!

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Erratum

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Page 5: The sentence on line 22 should read

This was closely followed by the discovery of sex in bacteria by Joshua Lederberg in 1946 and the recognition of the idea that each gene is a carrier of information for the production of an enzyme (the one gene–one enzyme hypothesis) proposed by Beadle and Tatum in 1941.