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A Short Note on the Versatile Power Mean

The power mean of two positive numbers (not necessarily distinct) denoted by $M_p(a, b)$ is defined as

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}, \quad p \neq 0.$$

This expression is of tremendous potential as you will soon see! Here are some special cases of interest:-

For $p = 1, M_1 = \frac{a+b}{2}$ gives the arithmetic mean of the numbers a and b.

For $p = -1, M_{-1} = \frac{2ab}{a+b}$ gives the harmonic mean of a and b.

For $p = 2$, we have $M_2 = [\frac{1}{2}(a^2 + b^2)]^{1/2}$ which statisticians refer to as the "root mean-square". If a and b represent deviations of two observations (say x and y) from their mean, the expression for M_2 represents standard deviation of x and y (we agree to take positive square-root only). Note that in this case, one of the numbers can be negative.

An illuminating result due to Kazarinoff (see [1]) is that $\lim_{p \to 0} M_p(a, b) = \sqrt{ab}$ so that one may arguably accept $M_0 = \sqrt{ab}$ giving the geometric mean of a and b!

Also of interest is the expression

$$M_{1/3} = \left(\frac{a^{1/3} + b^{1/3}}{2}\right)^3 \text{ for } p = 1/3$$

which physicists call as the Lorentz mean (sometimes also referred to as the Lorentz combination) and finds applications in areas such as the theory of equation of state of gases.

Keywords
Power mean, arithmetic mean, harmonic mean, root mean square, logarithmic mean, geometric mean.

Another interesting result about the power mean is that it is an increasing function of p, i.e. $M_p(a,b) < M_q(a,b)$ whenever $p < q$. This result is again credited to Kazarinoff [1]. It immediately follows from above that $M_{-1} < M_0 < M_1$ which leads to the well-known statistical inequality H.M. < G.M. < A.M. with the inequality becoming an equality when $a = b$.

Relationship between power mean and logarithmic mean:

Let a and b be two positive numbers, $a \neq b$. We define the logarithmic mean of a and b denoted by $L(a,b)$ as $L(a,b) = \frac{a-b}{\log_e^a - \log_e^b}$. Although not as versatile as the power mean, logarithmic mean also has useful applications in areas like heat transfer and fluid mechanics. Mitrinovic showed that $M_0 < L(a,b) < M_1, a \neq b$ (see [2] for the proof) which was bettered by Carlson [3] to $M_0 < L(a,b) < M_{1/2},\ a \neq b$. These results led to the logical question of finding the least number p and the greatest number q such that $M_p < L(a,b) < M_q$. Fortunately, the question stands settled with the required $p = 0$ and the required $q = 1/3$. A formal proof is beyond the scope of the present note. Interested readers are referred to the article by Tung-Po Lin (see [4]).

A question that interests me is whether power mean can be defined for n positive numbers. I would propose the formula, for n positive numbers $x_i, i = 1, 2, \ldots, n$,

$$M_p(x_1, x_2, \ldots, x_n) = \left(\frac{x_1^p + x_2^p + \ldots x_n^p}{n}\right)^{1/p},\ p \neq 0.$$

It is easy to see that $M_1 =$ A.M. and $M_{-1} =$ H.M. once again! $M_0 =$ G.M. would be acceptable provided one can show that

$$\lim_{p \to 0} M_p(x_1, x_2, \ldots, x_n) = (x_1 x_2 x_3 \ldots x_n)^{1/n}.$$

See Appendix for the proof.

Appendix

I shall prove a result more general than that of Kazarinoff

To prove:

$$\underset{p\to 0}{Lt}\left(\frac{\sum_{i=1}^{k} x_i^p}{k}\right)^{1/p} = \left(\prod_{i=1}^{k} x_i\right)^{1/k}, \quad x_i > 0,\ i = 1, 2, \ldots, k$$

Proof:

$$\text{L.H.S.} = \underset{p\to 0}{Lt}\left(\sum_{i=1}^{k} x_i^p\right)^{1/p} \cdot \frac{1}{k^{1/p}}$$

$$\text{Let } y = \left(\sum_{i=1}^{k} x_i^p\right)^{1/p} \cdot \frac{1}{k^{1/p}}$$

$$\Rightarrow \log y = \frac{1}{p}\log\left(\sum_{i=1}^{k} x_i^p\right) - \log k^{1/p}$$

$$= \frac{1}{p}\left[\log\left(\sum_{i=1}^{k} x_i^p\right) - \log k\right]$$

$$\underset{p\to 0}{Lt}\ \log y = \underset{p\to 0}{Lt}\left[\frac{\log(\sum_{i=1}^{k} x_i^p) - \log k}{p}\right] \qquad (1)$$

Since $x_i > 0, i = 1, 2, \ldots, k$, hence $x_i^0 = 1, i = 1, 2, \ldots, k$.

Consequently, R.H.S. takes the 0/0 from on taking the limit as $p \to 0$. Applying L'Hospital's rule,

$$\underset{p\to 0}{Lt}\ \log y = \underset{p\to 0}{Lt}\ \frac{\sum_{i=1}^{k} x_i^p \log x_i}{\sum_{i=1}^{k} x_i^p}$$

$$= \frac{\sum_{i=1}^{k} \log x_i}{k}$$

$$= \log\left(\prod_{i=1}^{k} x_i\right)^{1/k} \qquad (2)$$

Now, it follows from the continuity of the logarithmic function that

$$\underset{p\to 0}{Lt}\ \log y = \log \underset{p\to 0}{Lt}\ y \qquad (3)$$

From (2) and (3), we have

$$\log \underset{p\to 0}{Lt}\ y = \log \left(\prod_{i=1}^{k} x_i \right)^{1/k}$$

$$\Rightarrow \underset{p\to 0}{Lt}\ y = \left(\prod_{i=1}^{k} x_i \right)^{1/k}$$

$$\text{Or,}\quad \underset{p\to 0}{Lt} \left(\frac{\sum_{i=1}^{k} x_i^p}{k} \right)^{1/p} = \left(\prod_{i=1}^{k} x_i \right)^{1/k}, \text{ as desired.}$$

For $k = 2$ we get Kazarinoff's result.

Suggested Reading

[1] N D Kazarinoff, *Analytic Inequalities*, Holt, New York, pp.63–64, 1961.

[2] D S Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Berlin, pp. 272–273, 1970.

[3] B C Carlson, *Amer. Math. Monthly*, Vol.79, pp.615–618, 1972.

[4] Tung-Po Lin, *Amer. Math. Monthly*, Vol.81, pp.879–883, 1974.

Erratum

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