In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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**n-Dimensional Cube and Simplex: A Glimpse into the Concept of Multi-Dimensional Spaces**

A close study of a square and a cube helps in extrapolating the concepts of vertex, edge and face to higher dimensions. Recurrence relationships have been heuristically arrived at, with the help of which high school students could be taught to calculate iteratively the number of vertices, edges, faces, etc of cube-like objects and simplexes of higher dimensions.

1. Introduction

Human beings have no difficulty in understanding what a straight line is, what a square is or what a cube is, because of their ability to see those objects to be what they are. A school boy will be able to say how many edges and vertices there are in a square, which is a 2-dimensional object, or how many faces, edges and vertices there are in a cube, which is a 3-dimensional object. However it is not possible for us to say with the same ease how many vertices, edges or faces there are in a 4-dimensional cube, as we have no way of visualizing a 4-dimesional cube, whatever it may mean. Our sensory perception is limited to only the three dimensions of space into which we are born.

Mathematics, which is a magnificent edifice built on the foundations of human reasoning, has a way of transcending the limitations of
sensory perception. In geometry, among other things, we study the properties of a point, a straight line, a square and a cube. If we correctly extrapolate the properties of these simple objects of perception, we can indeed learn a lot about similar objects of 4, 5 or higher dimensions. This article attempts to explain how this can be done heuristically.

2. We shall now look at a certain set of properties of a straight line, a square and a cube in an integrated manner.

1. A straight line is a 1-dimensional object which is bounded by 2 end points. The infinitely long straight line which contains the straight line is a 1-dimensional space.

2. A square is a 2-dimensional object which is bounded by 4 vertices and 4 edges. The infinite plane which contains the square is a 2-dimensional space. A square can be generated by moving a straight line parallel to itself in the second dimension of the 2-dimensional space such that the distance moved is equal to the length of the straight line.

3. A cube is a 3-dimensional object which is bounded by 8 vertices, 12 edges and 6 faces. The infinite space which contains the cube is a 3-dimensional space. A cube can be generated by moving a square parallel to itself in the third dimension such that the distance moved is equal to the length of the edges of the square.

In the above description you will notice that a cube, a 3-D (hereafter ‘dimensional’ will be abbreviated by ‘-D’) object is bounded by vertices (0-D object), edges (1-D object) and faces (2-D object). Similarly a square, a 2-D object, is bounded by vertices (0-D object) and edges (1-D object) and a straight line, a 1-D object is bounded by end points (0-D object). It appears logical to extrapolate that an n-D object will be bounded by objects of (n – 1)-D, (n – 2)-D, --- 3-D, 2-D, 1-D and 0-D. Beyond two dimensions it is impracticable to name such objects by individual names like vertex, edge, face, etc., mathematicians call them by a common name ‘hyper plane’. Thus a 4-D cube will be bounded by vertices, edges, faces and 3-D hyper planes.

It is not possible for us to say with ease how many vertices, edges or faces there are in a 4-dimensional cube, as we have no way of visualizing a 4-dimesional cube.

If we correctly extrapolate the properties of simple objects of perception, we can indeed learn a lot about similar objects of 4, 5 or higher dimensions.
In this article, however, for convenience, we shall call vertex or an end point by ‘0-D bound’, an edge by ‘1-D bound’, a face by ‘2-D bound’ and an n-D hyper plane by ‘n-D bound’. Using this terminology we shall expand the above descriptions as follows:

1. A point is a 0-D object in a 0-D space. It has 1 0-D bound (the point itself).

2. A straight line is a 1-D object in a 1-D space. It has 2 0-D bounds and 1 1-D bound (the straight line itself). It can be generated by moving a 0-D object to a defined distance in the 1st dimension. Note that 1 0-D bound when moved, generates another 0-D bound and 1 1-D object. The movement thus results in 1 1-D object having 2 0-D bounds. (Figure 1)

3. A square is a 2-D object in a 2-D space. It has 4 0-D bounds, 4 1-D bounds and 1 2-D bound. It can be generated by moving a straight line, a 1-D object, parallel to itself in the 2nd dimension by a distance equal to the length of its (of 1-D object) 1-D bound. Each 0-D bound of the 1-D object generates another 0-D bound and 1 1-D bound. The 1-D bound of 1-D object generates another 1-D bound and 1 2-D object. The movement thus results in 1 2-D object having 4 0-D bounds and 4 1-D bounds.

4. A cube is a 3-D object in a 3-D space. It can be generated by moving a square, a 2-D object, parallel to itself in the 3rd dimension by a distance equal to the length of its 1-D bound. Each 0-D bound of the 2-D object generates another 0-D bound.

**Figure 1. Generation of straight line, square and cube from point, straight line and square.**
and 1 1-D bound. Each 1-D bound of 2-D object generates another 1-D bound and 1 2-D bound. The 2-D bound of 2-D object generates another 2-D bound and 1 3-D object. The movement thus results in 1 3-D object having 8 0-D bounds, 12 1-D bounds, and 6 2-D bounds.

In the above, use of words such as 0-D bound, 1-D bound and 2-D bound may appear clumsy at first glance. But a closer look will show that it reveals the recursive and iterative manner in which the bounds of an $n$-dimensional cube are related to the bounds of an $(n-1)$ dimensional cube.

We shall represent the same results in a tabular form (Table 1), where $B_{j,k}$ gives the number of $k$-D bounds of a $j$-D cube. For want of a specific word, we have used the word ‘cube’ here to refer to cube-like objects of higher dimensions. Table 1 lists out the value of $B_{j,k}$ for values $j = 0$ to $j = 5$.

We can now write a general recurrence formula as follows:

$$B_{n,m} = B_{n-1, m-1} + 2*B_{n-1, m}$$  \hspace{1cm} (1)

where $B_{n,m}$ is the number of $m$-dimensional bounds in an $n$-dimensional space cube, given that $n \geq m \geq 0$ and $B_{j,k} = 1$ when $j = k$.

One way of seeing why (1) is true is to examine the case $m = 1$. Then (1) reads: $B_{n,1} = B_{n-1,0} + 2*B_{n-1, 1}$. This is true because each edge of a $(n-1)$-D cube yields two edges of the $n$-D cube as a result of the movement through the new dimension (we get two ‘copies’ of each edge), and each vertex of the $(n-1)$-D cube yields one edge of the $n$-D cube. Similar reasoning works for $m > 1$.

<table>
<thead>
<tr>
<th>N</th>
<th>$B_{0,0}$=1</th>
<th>$B_{1,1}$</th>
<th>$B_{2,2}$</th>
<th>$B_{3,3}$</th>
<th>$B_{4,4}$</th>
<th>$B_{5,5}$</th>
<th>$B_{n-1,0}$</th>
<th>$B_{n-1,1}$</th>
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<td></td>
<td>B2,0=2</td>
<td>B1,0=2</td>
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<tr>
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<td>B1,1=1</td>
<td>B1,0=2</td>
<td>B2,1=4</td>
<td>B3,2=8</td>
<td>B4,3=16</td>
<td>B5,4=32</td>
<td>B2,1=4</td>
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<td></td>
<td></td>
<td>B6,5=32</td>
<td>B5,4=32</td>
</tr>
</tbody>
</table>
We also see that the following equations hold good:

\[ B_{n,n-1} = 2n \]  
\[ B_{n,0} = 2^n \]  
\[ \sum_{k=0}^{k=n} B_{n,k} = 3^n \]

It needs to be noted that the above is equally applicable to a rectangle, rectangular parallelepiped and similar objects of higher dimensions.

3. Similarly we can work out for the simplexes of various dimensions. A straight line having two 0-D bounds is a simplex of 1-dimension. A triangle is a simplex of 2-dimensions. A tetrahedron is a simplex of 3-dimensions. A triangle is generated by joining the 0-D bounds of a 1-D simplex to another 0-D object not contained in the 1-D space. A tetrahedron is generated by joining the 0-D bounds of a 2-D simplex to a 0-D object not contained in the 2-D space (Figure 2).

The number of 0-D bounds of 2-D simplex is 3, one more than the number of 0-D bounds of 1-D simplex. The number of 1-D bounds of 2-D simplex is 3, equal to the sum of the number of 0-D bounds and the number of 1-D bounds of the 1-D simplex (2+1=3).

The number of 0-D bounds of 3-D simplex is 4. The number of 1-D bounds of 3-D simplex is 6, the sum of the number of 0-D bounds and the number of 1-D bounds of the 2-D simplex (3+3=6). The number of 2-D bounds of 3-D simplex is 4, the sum of the number of 1-D bounds and 2-D bounds of the 2-D simplex (3+1=4).

We can rewrite these results as in Table 2; we have included the 4-D and 5-D simplexes too.

Now we can write a general formula as follows:

\[ B'_{n,m} = B'_{n-1,m-1} + B'_{n-1,m} \]

where \( B'_{n,m} \) is the number of \( m \)-D bounds of an \( n \)-D simplex given that \( n \geq m \geq 0 \) and \( B'_{j,k} = 1 \) when \( j = k \).
As earlier, it is best to start with the case \( m = 1 \). Then (5) reads:
\[ B'_{n,1} = B'_{n-1,0} + B'_{n-1,1}. \]
This is true because when we join each vertex of a \((n-1)\)-D simplex to a point in the next dimension, we get one new edge for each vertex, so the number of new edges introduced is the same as the number of vertices in the \((n-1)\)-D simplex. Similar reasoning works for \( m > 1 \).

One cannot miss to note that the above table is a truncated Pascal’s triangle.

We also see that the following equations hold good:
\[ B'_{n,-1} = B'_{n,0} = n+1 \]  \hspace{2cm} (6)
\[ \sum_{k=0}^{n} B'_{n,k} = 2^{n+1} - 1 \]  \hspace{2cm} (7)

We can also work out the details in a similar manner for polyhedrons and pyramids of higher dimensions.

4. Conclusion

This article attempts to explain how a glimpse of the concept of dimensions higher than three can be given to even high school students through an iterative procedure to calculate the number of vertices, edges, faces, etc., of an \( n \)-dimensional cube and an \( n \)-dimensional simplex.

### Table 2. Number of vertices, edges, faces, etc., of simplexes of 0–5 dimensions.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( B'_{n,0} )</th>
<th>( B'_{n,-1} )</th>
<th>( B'_{n,-2} )</th>
<th>( B'_{n,-3} )</th>
<th>( B'_{n,-4} )</th>
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</tbody>
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Suggested Reading