On Sums of Powers of Integers

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For non-negative integers k, n, let $P_k(n)$ denote the sum

$$1^k + 2^k + \cdots + (n-1)^k + n^k.$$

We show by two different means that if $k \geq 3$ and odd, then $n^2(n+1)^2$ iss a factor of the polynomial $P_k(n)$; and if $k \geq 2$ and even, then $n(n+1)(2n+1)$ is a factor of the polynomial $P_k(n)$. We also derive a relatively unknown result first obtained by Johann Faulhaber in the 17th century.

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1. Introduction

Write $P_k(n)$ for the sum $1^k+2^k+\cdots+(n-1)^k+n^k$, where k and n denote non-negative integers; then $P_k(1) = 1$ for all $k \in \mathbb{N}$. We may take $P_k(0) = 0$ for all k, by using the convention that an empty sum equals 0. Mathematical folklore tells us that Gauss as a young boy found (only in effect; in the actual incident, n was 100) a formula for $P_1(n)$, by the simple device of adding the first term to the last term, the second term to the second-last term, and so on. So, from

$$P_1(n) = 1 + 2 + 3 + \cdots + (n-1) + n,$$

he got:

$$\begin{aligned} 2P_1(n) &= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} \\ &= n(n+1), \end{aligned}$$

giving $P_1(n) = \frac{1}{2}n(n+1)$. This elegant device does not work for the sums of the squares, cubes, etc., but there are other ways of finding these sums: guessing the

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formulas, then proving them using mathematical induction; or by using a technique called *telescoping*, of which the prototypical example is the summation

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n} = \sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$
$$= 1-\frac{1}{n}.$$

Jakob Bernoulli made use of this idea in proving the divergence of the harmonic series, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$. By using telescoping, we can arrive at simple formulas for the $P_k(n)$. (But Bernoulli used yet other ideas to find formulas for these sums.)

Our intention in this article is to prove some pretty factorizability properties of these functions. On asking a computer algebra system like *Mathematica* to give expressions for the $P_k(n)$ in factorized form, we get the display shown in *Table* 1. (The *Mathematica* commands needed to accomplish this are given at the end of the article in *Box* 1.) The following facts strike the eye almost immediately:

- $n^2(n+1)^2$ is a factor of the polynomial $P_k(n)$ for $k = 3, 5, 7, 9$.
- $n(n+1)(2n+1)$ is a factor of the polynomial $P_k(n)$ for $k = 2, 4, 6, 8, 10$.

We shall show that these properties persist for higher values of k.

2. Proofs Using Induction

Consider the following relation given by the binomial theorem ($k \geq 0$):

$$
\begin{aligned}
P_1(n) &= \tfrac{1}{2}n(n+1),\\
P_2(n) &= \tfrac{1}{6}n(n+1)(2n+1),\\
P_3(n) &= \tfrac{1}{4}n^2(n+1)^2,\\
P_4(n) &= \tfrac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right),\\
P_5(n) &= \tfrac{1}{12}n^2(n+1)^2\left(2n^2+2n-1\right),\\
P_6(n) &= \tfrac{1}{42}n(n+1)(2n+1)\left(3n^4+6n^3-3n+1\right),\\
P_7(n) &= \tfrac{1}{24}n^2(n+1)^2\left(3n^4+6n^3-n^2-4n+2\right),\\
P_8(n) &= \tfrac{1}{90}n(n+1)(2n+1)\left(5n^6+15n^5+5n^4-15n^3-n^2+9n-3\right),\\
P_9(n) &= \tfrac{1}{20}n^2(n+1)^2\left(n^2+n-1\right)\left(2n^4+4n^3-n^2-3n+3\right),\\
P_{10}(n) &= \tfrac{1}{66}n(n+1)(2n+1)\left(n^2+n-1\right)\left(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5\right).
\end{aligned}
$$

Table 1. The polynomials $P_1(n)$, $P_2(n)$, ... , $P_{10}(n)$, in factorized form.

$$(n+1)^{k+1} - n^{k+1} = (k+1)n^k + \binom{k+1}{2} n^{k-1} + \binom{k+1}{3} n^{k-2} + \cdots + \binom{k+1}{k} n + 1.$$

If we replace n in this equation successively by $n-1$, $n-2$, ..., 2, 1 and then add the corresponding sides of all the equations, the left side telescopes, and we get:

$$(n+1)^{k+1} - 1 = (k+1)P_k(n) + \binom{k+1}{2} P_{k-1}(n) + \cdots + \binom{k+1}{k} P_1(n) + n.$$

The function P_k introduced at the start is defined only for non-negative integer arguments. But now – having shown that it is polynomial in form – we can extend its definition to all the real numbers.

This yields, on transposition:

$$(k+1)P_k(n) = (n+1)^{k+1} - 1 - \sum_{r=2}^{k+1} \binom{k+1}{r} P_{k+1-r}(n). \quad (1)$$

Since $P_0(n) = n$, it follows inductively from this equation that $P_k(n)$ is a polynomial in n of degree $k+1$ for all $k \in \mathbb{N}$, with leading coefficient $\frac{1}{k+1}$.

An important conceptual change may now be made. The function P_k introduced at the start is defined only for non-negative integer arguments. But now – having shown that it is polynomial in form – we can extend its definition to all the real numbers. This allows us to compute derivatives, non-integer roots, etc., of the P_k, which otherwise lack meaning. To signal this change, we write $Q_k(x)$ where earlier we wrote $P_k(n)$, and define the polynomials Q_k recursively for $k \in \mathbb{N}$ as follows:

$$Q_0(x) = x, \quad (k+1)Q_k(x) = (x+1)^{k+1} - 1 - \sum_{r=2}^{k+1} \binom{k+1}{r} Q_{k+1-r}(x). \quad (2)$$

Then $Q_k(x)$ is a polynomial in x of degree $k+1$ for all non-negative integers k, with leading coefficient $\frac{1}{k+1}$. The claim now is that $x^2(x+1)^2$ is a factor of $Q_k(x)$ for $k \geq 3$, odd; and $x(x+1)(2x+1)$ is a factor of $Q_k(x)$ for $k \geq 2$, even.

From equation (2) we get, by differentiation with respect to x:

$$(k+1)Q'_k(x) = (k+1)(x+1)^k - \sum_{r=2}^{k+1} \binom{k+1}{r} Q'_{k+1-r}(x).$$

From the same equation we also get:

$$kQ_{k-1}(x) = (x+1)^k - 1 - \sum_{r=2}^{k} \binom{k}{r} Q_{k-r}(x).$$

The preceding two equations, which are true for all $k \in \mathbb{N}$, yield the following:

$$Q'_k(x) - kQ_{k-1}(x) =$$

$$1 - \sum_{r=2}^{k+1} \left[\frac{1}{k+1} \binom{k+1}{r} Q'_{k+1-r}(x) - \binom{k}{r} Q_{k-r}(x) \right].$$

The general bracketed term here is

$$\frac{1}{k+1} \binom{k+1}{r} Q'_{k+1-r}(x) - \binom{k}{r} Q_{k-r}(x) =$$

$$\binom{k}{r} \left[\frac{1}{k+1-r} Q'_{k+1-r}(x) - Q_{k-r}(x) \right].$$

Since $Q'_1(x) - Q_0(x) = \frac{1}{2}$, which is independent of x, it follows inductively that $Q'_k(x) - kQ_{k-1}(x)$ depends only on k and is independent of x, for all $k \in \mathbb{N}$. That is,

$$Q'_k(x) - kQ_{k-1}(x) = \text{ constant } (k) \quad (\text{all } k \in \mathbb{N}). \quad (3)$$

Proofs of the Factorizability Properties: We now show the factorizability properties claimed earlier, that $x^2(x+1)^2$ is a factor of $Q_k(x)$ if k is odd and greater than or equal to 3; and $x(x+1)(2x+1)$ is a factor of $Q_k(x)$ if k is even and greater than or equal to 2. These will be achieved by showing the following:

- $Q_k(0) = 0 = Q'_k(0)$ and $Q_k(-1) = 0 = Q'_k(-1)$ for $k \geq 3$, odd;
- $Q_k(0) = 0$, $Q_k(-1) = 0$ and $Q_k(-\frac{1}{2}) = 0$ for $k \geq 2$, even.

Using the binomial theorem to expand $(n+1)^{k+1}$ and $(n-1)^{k+1}$, we get

Equation (3') yields a recursive relation for $P_k(n)$ in terms of $P_{k-2}(n)$, $P_{k-4}(n)$, ...

$$(n+1)^{k+1} - (n-1)^{k+1} =$$

$$2\left[\binom{k+1}{1} n^k + \binom{k+1}{3} n^{k-2} + \cdots\right].$$

If we replace n in this equation successively by $n-1$, $n-2$, ..., 2, 1 and add the corresponding sides of all the equations, the left side *almost* telescopes, and we get:

$$(n+1)^{k+1} + n^{k+1} - 1 =$$

$$2\left[(k+1)P_k(n) + \binom{k+1}{3} P_{k-2}(n) + \cdots\right]. \quad (3')$$

This yields a recursive relation for $P_k(n)$ in terms of $P_{k-2}(n)$, $P_{k-4}(n)$, ... Though we have made use of the fact that n is a non-negative integer in this derivation, we can extend the same recursion to the polynomials $Q_k(x)$, and we get, for all $k \in \mathbb{N}$:

$$2(k+1)Q_k(x) = (x+1)^{k+1} + x^{k+1} - 1$$

$$-2\left[\binom{k+1}{3} Q_{k-2}(x) + \binom{k+1}{5} Q_{k-4}(x) + \cdots\right]. \quad (4)$$

We now consider separately the cases when k is odd and when k is even.

If k is odd and at least 3, then the last term in equation (4) is $\binom{k+1}{k} Q_1(x)$ which is equal to $(k+1)\frac{1}{2}x(x+1)$, so:

$$2(k+1)Q_k(x) = (x+1)^{k+1} + x^{k+1} - 1$$

$$-2\left[\binom{k+1}{3} Q_{k-2}(x) + \cdots + \binom{k+1}{k-2} Q_3(x)\right] -$$

$$(k+1)x(x+1).$$

From this we get, inductively, $2(k+1)Q_k(0) = 1+0-1-2(0)-0 = 0$. Next, by differentiation we get (inductively, again),

$$2(k+1)Q'_k(x) = (k+1)\left[(x+1)^k + x^k\right]$$

$$-\ 2\left[\binom{k+1}{3} Q'_{k-2}(x) + \cdots + \binom{k+1}{k-2} Q'_3(x)\right] -$$

$$(k+1)(2x+1).$$

From this we deduce that

$$2(k+1)Q'_k(0) \ = \ (k+1)(1) - 2(0) - (k+1)(1) = 0.$$

It follows that x^2 is a factor of $Q_k(x)$ for $k \geq 3$, odd.

From the above two equations we also get:

$$2(k+1)Q_k(-1) \ = \ 0+(-1)^{k+1}-1-2(0)-(k+1)(0) \ = \ 0,$$

and

$$2(k+1)Q'_k(-1) \ = \ (k+1)(-1)^k-2(0)-(k+1)(-1) \ = \ 0.$$

It follows that $(x+1)^2$ is a factor of $Q_k(x)$ for $k \geq 3$, odd. Therefore, $x^2(x+1)^2$ is a factor of $Q_k(x)$ for all such k.

If k is even and at least 2, then the last term in equation (4) is $\binom{k+1}{k+1} Q_0(x) = x$, so we get:

$$2(k+1)Q_k(x) = (x+1)^{k+1} + x^{k+1} - 1$$

$$-2\left[\binom{k+1}{3} Q_{k-2}(x) + \binom{k+1}{5} Q_{k-4}(x) + \cdots +\right.$$

$$\left.\binom{k+1}{k-1} Q_2(x)\right] - 2x.$$

Given a function f from the non-negative integers $\mathbb{N} \geq 0$ into the set of real numbers $\mathbb{R}$, can we extend the domain of f to $\mathbb{R}$ (or to $\mathbb{R} \geq 0$) in a natural way so that the resulting function is differentiable and otherwise 'nice'?

This yields (arguing inductively, as earlier):

$$\begin{aligned} 2(k+1)Q_k(0) &= 1+0-1-2(0)-0=0, \\ 2(k+1)Q_k(-1) &= 0+(-1)^{k+1}-1-2(0)+2= \\ &\qquad -2+2=0, \\ 2(k+1)Q_k\left(-\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^{k+1}+\left(-\frac{1}{2}\right)^{k+1}-1 \\ &\qquad -2(0)+1=0. \end{aligned}$$

It follows that $x(x+1)(2x+1)$ is a factor of $Q_k(x)$ for $k \geq 2$, even. □

Remark. The point made earlier about extending the domain of definition of the P_k from the non-negative integers to the real numbers has relevance for other number theoretic functions. One may pose the following general question:

> Given a function f from the non-negative integers $\mathbb{N}_{\geq 0}$ into the set of real numbers $\mathbb{R}$, can we extend the domain of f to $\mathbb{R}$ (or to $\mathbb{R}_{\geq 0}$) in a natural way so that the resulting function is differentiable and otherwise 'nice'?

Of course, the answer will depend on f. A famous instance where it has been done is the extension (achieved by Euler; who else?) of the factorial function $f(n) = n!$ to the Gamma function, by cleverly exploiting the identity $\int_0^\infty t^n e^{-t}\, dt = n!$ (which is valid for all non-negative integers n).

3. A Cookbook Recipé

The properties listed above allow the formulation of a neat two-step algorithm for generating the polynomials P_k recursively, starting with $P_0(n) = n$. The rule is:

1) Compute the integral $\int_0^n k \cdot P_{k-1}(x)\, dx$.

2) Add a suitable multiple of n to the answer so that the algebraic sum of the coefficients of the resulting polynomial is 1. The answer is $P_k(n)$.

Example: Starting with $P_0(n) = n$ we first get $\int_0^n P_0(x)\,dx = \frac{1}{2}n^2$; to this we add $\frac{1}{2}n$, because $\frac{1}{2} + \frac{1}{2} = 1$, and we get $P_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$.

Next, we have $\int_0^n 2P_1(x)\,dx = \frac{1}{3}n^3 + \frac{1}{2}n^2$; to this we add $\frac{1}{6}n$, because $\frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$, and we get $P_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

And so on .... This is as neat a cookbook recipé as anyone may want!

4. Bernoulli Polynomials

We now follow a completely different approach and use generating functions to prove the claims made about the factors of the polynomials Q_k. More specifically, we use the *Bernoulli polynomials* which crop up repeatedly in different parts of mathematics, rather like the Fibonacci numbers or the Catalan numbers. In this section we introduce these polynomials.

The Bernoulli polynomials may be defined in many ways; there is one approach based on determinants! The most direct way is to expand the following function,

$$G(0, x) = 1, \qquad G(z, x) = \frac{z \cdot e^{zx}}{e^z - 1} \quad \text{for} \quad z \neq 0,$$

as a power series in z. We write

$$\frac{z \cdot e^{zx}}{e^z - 1} = \sum_{k \geq 0} B_k(x) \frac{z^k}{k!}; \tag{5}$$

then $B_k(x)$ is defined to be the k^{th} Bernoulli polynomial. We clearly have

$$B_k(x) = \left. \frac{\partial^k G(z, x)}{\partial z^k} \right|_{z \to 0}.$$

We use the *Bernoulli polynomials* which crop up repeatedly in different parts of mathematics, rather like the Fibonacci numbers or the Catalan numbers.

Table 2. The polynomials $B_1(x), B_2(x), \ldots, B_6(x)$.

$$
\begin{aligned}
B_1(x) &= x - \tfrac{1}{2},\\
B_2(x) &= x^2 - x + \tfrac{1}{6},\\
B_3(x) &= x^3 - \tfrac{3x^2}{2} + \tfrac{x}{2},\\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30},\\
B_5(x) &= x^5 - \tfrac{5x^4}{2} + \tfrac{5x^3}{3} - \tfrac{x}{6},\\
B_6(x) &= x^6 - 3x^5 + \tfrac{5x^4}{2} - \tfrac{x^2}{2} + \tfrac{1}{42}.
\end{aligned}
$$

Using this, or any of several other methods, the Bernoulli polynomials may be computed. *Table* 2 displays the first few of them.

Now observe that

$$\frac{\partial G(z,x)}{\partial x} = \frac{z^2 e^{zx}}{e^z - 1} = \sum_{k\geq 0} B_k(x)\frac{z^{k+1}}{k!}.$$

So the coefficient of z^k in $\frac{\partial G(z,x)}{\partial x}$ is $\frac{B_{k-1}(x)}{(k-1)!}$. On the other hand,

$$\frac{\partial}{\partial x}\left(\sum_{k\geq 0} B_k(x)\frac{z^k}{k!}\right) = \sum_{k\geq 0} B_k'(x)\frac{z^k}{k!},$$

and in the expression on the right, the coefficient of z^k is $\frac{B_k'(x)}{k!}$.

Comparing the two results we deduce that

$$B_k'(x) = kB_{k-1}(x).$$

This relation proves to be of great importance. Another such relation is:

$$\int_0^1 B_k(x)\,dx = 0 \quad \text{for all} \quad k \geq 1.$$

Conditions (6) uniquely fix the B_k.

The reader is invited to supply the proof of this, using the defining condition (5).

Alternative Definition. As noted earlier, the Bernoulli polynomials may be defined in different ways; we now choose to define them through their properties. That is, we define them recursively using the following:

$$B_0(x) = 1, \qquad B_k'(x) = kB_{k-1}(x),$$
$$\int_0^1 B_k(x)\,dx = 0 \quad \text{for} \quad k \geq 1. \tag{6}$$

These conditions uniquely fix the B_k. First we shall show that we get the same polynomials as earlier. Let $H(z,x)$ be given by the sum

$$H(z,x) = \sum_{k\geq 0} B_k(x)\frac{z^k}{k!}.$$

Then we have:

$$\begin{aligned} zH(z,x) &= \sum_{k\geq 0} B_k(x)\frac{z^{k+1}}{k!} = \sum_{k\geq 0}(k+1)B_k(x)\frac{z^{k+1}}{(k+1)!} \\ &= \sum_{k\geq 0} B_{k+1}'(x)\frac{z^{k+1}}{(k+1)!} = \frac{\partial H}{\partial x}, \\ \therefore\ H(z,x) &= f(z)\cdot e^{zx} \quad \text{for some function} \quad f. \end{aligned}$$

The defining conditions imply that $\int_0^1 H(z,x)\,dx = 1$, so $f(z)\cdot\int_0^1 e^{zx}\,dx = 1$. This yields:

$$f(z) = \frac{z}{e^z - 1}, \qquad \therefore\ H(z,x) = \frac{z\cdot e^{zx}}{e^z - 1} = G(z,x).$$

So the two definitions are equivalent to each other.

Some Properties of the Bernoulli Polynomials: We now show some nice properties of the B_k which prove to be crucial.

Property 1: $B_k(1) = B_k(0)$ for $k \geq 2$.

For, since $B'_k(x) = kB_{k-1}(x)$ for $k \geq 1$, we have $\int_0^1 B'_k(x)\,dx = \int_0^1 kB_{k-1}\,dx$. But:

$$\int_0^1 B'_k(x)\,dx = B_k(x)\Big]_{x=0}^{x=1} = B_k(1) - B_k(0),$$

$$\text{and} \quad \int_0^1 kB_{k-1}(x)\,dx = 0 \text{ for } k-1 \geq 1, \text{i.e., for } k \geq 2.$$

It follows that $B_k(1) = B_k(0)$ for $k \geq 2$.

Property 2: $B_k(x+1) - B_k(x) = kx^{k-1}$ for all $k \geq 1$.

To start the induction: $B_1(x+1) - B_1(x) = (x + \frac{1}{2}) - (x - \frac{1}{2}) = 1 = 1 \cdot x^0$.

Next, the derivative of $B_k(x+1) - B_k(x)$ is

$$\begin{aligned} B'_k(x+1) - B'_k(x) &= k\left[B_{k-1}(x+1) - B_{k-1}(x)\right] \\ &= k \cdot (k-1)x^{k-2} \quad \text{(inductively)}, \end{aligned}$$

so we get, by integration, $B_k(x+1) - B_k(x) = kx^{k-1} +$ constant.

The substitution $x = 0$ shows that the constant must be 0, and we get the stated result.

Connection with Sums of Powers: At long last we begin to see the connection between the Bernoulli polynomials and the sums of powers of integers. For, if n is any positive integer, then we get from Property 2 (above):

At long last we begin to see the connection between the Bernoulli polynomials and the sums of powers of integers.

$$B_{k+1}(n+1) - B_{k+1}(n) = (k+1)n^k.$$

Successively replacing n by $n-1$, $n-2$, $\ldots$, 1 and adding the corresponding sides of these equations, we find that the left side telescopes, and we get:

$$P_k(n) = \frac{B_{k+1}(n+1) - B_{k+1}(1)}{k+1}. \tag{7}$$

Following the identification we made between the functions P_k and Q_k, we see that we also have the identity

$$Q_k(x) = \frac{B_{k+1}(x+1) - B_{k+1}(1)}{k+1}. \tag{8}$$

Property 3. $B_k(1-x) = (-1)^k B_k(x)$ for all $k \geq 1$.

To prove this we fall back on the original definition, equation (5). This yields, via the substitution $x \mapsto 1-x$:

$$\begin{aligned} G(z, 1-x) &= \frac{z\, e^{z(1-x)}}{e^z - 1} = \frac{z\, e^z}{e^z - 1}\, e^{-zx} \\ &= \frac{z}{1 - e^{-z}}\, e^{-zx} = \frac{(-z)}{e^{-z} - 1}\, e^{(-z)x} \\ &= G(-z, x), \\ \therefore \ \sum_{k \geq 0} B_k(1-x) \frac{z^k}{k!} &= \sum_{k \geq 0} B_k(x) \frac{(-z)^k}{k!}. \end{aligned}$$

Property 3 follows by comparing coefficients of z^k on both sides.

Corollary: $B_k\left(\frac{1}{2}\right) = 0$ for $k \geq 1$, odd.

Property 4. $B_{2k+1}(0) = 0$ for all $k \geq 1$.

To see why, we use equation (5) once again. Adding $\frac{z}{2}$ to $G(z, 0)$, we get:

$$\begin{aligned} G(z,0) + \frac{z}{2} &= \frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\left(\frac{2}{e^z - 1} + 1\right) \\ &= \frac{z}{2}\left(\frac{e^z + 1}{e^z - 1}\right) = \frac{z}{2}\left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}\right). \end{aligned}$$

This shows that $G(z, 0) + \frac{z}{2}$ is an *even* function of z, so $B_{2k+1}(0) = 0$ for all $k \geq 1$. (Note, however, that

From Property 5, we deduce that $B_k(x)$ is a monic polynomial with degree k.

$B_1(0) \neq 0$.) Using an earlier result, we deduce that we also have $B_{2k+1}(1) = 0$ for all $k \geq 1$.

Corollary: Recalling equation (7) we see that

$$\text{If } k \geq 2 \text{ and is even, then } P_k(n) = \frac{B_{k+1}(n+1)}{k+1}. \quad (9)$$

A further corollary may now be obtained; if $k \geq 2$ and is even, then

$$P_k\left(-\frac{1}{2}\right) = \frac{B_{k+1}\left(\frac{1}{2}\right)}{k+1} = 0.$$

Property 5. $B_k(x) = x^k - \frac{1}{k+1}\sum_{r=0}^{k-1} B_r(x)$.

This may be shown by examining the equality

$$z \cdot e^{zx} = (e^z - 1) \cdot \left(\sum_{k \geq 0} B_k(x)\frac{z^k}{k!}\right),$$

and comparing the coefficients of z^{k+1} on the two sides. From this property we deduce that $B_k(x)$ is a monic polynomial with degree k.

5. Proving the Factorizability Properties

We consider separately the cases when k is even and when k is odd.

k ≥ 2, even: In this case, we have:

$$\begin{aligned} P_k(0) &= \frac{B_{k+1}(1) - B_{k+1}(1)}{k+1} = 0; \\ P_k(-1) &= \frac{B_{k+1}(0) - B_{k+1}(1)}{k+1} = 0 \\ &\qquad \text{(Property 1, proved earlier)}; \\ P_k\left(-\frac{1}{2}\right) &= 0 \quad \text{(proved above)}. \end{aligned}$$

From these we conclude that $n(n+1)(2n+1)$ is a factor of $P_k(n)$.

k ≥ 3, odd: In this case:

$$\begin{aligned} P_k(-1) &= 0, \quad \text{as above;} \\ P'_k(-1) &= \frac{B'_{k+1}(0)}{k+1} = B_k(0) = 0, \\ P_k(0) &= 0, \\ P'_k(0) &= \frac{B'_{k+1}(1)}{k+1} = B_k(1) = B_k(0) = 0. \end{aligned}$$

From these we conclude that $n^2(n+1)^2$ is a factor of $P_k(n)$.

6. Faulhaber's Findings

If we examine *Table* 1 carefully we find to our astonishment that if k is odd, then $P_k(n)$ is actually a polynomial in the quantity $n(n+1)$. Write N for $n(n+1)$; then the polynomials are as displayed in *Table* 3.

For even values of k there is a less obvious relation; we find that $P_k(n)$ is equal to $(2n+1)$ times a polynomial in $n(n+1)$. The relations are shown in *Table* 4.

These relations were first discovered by Johann Faulhaber (1580–1635), an early algebraist who was a close friend of both Johannes Kepler and René Descartes. In keeping with the style of his time, he simply stated the formulas (that too, encrypted in a medieval code!), and

$$\begin{aligned} P_1(n) &= \tfrac{1}{2}N, \\ P_3(n) &= \tfrac{1}{4}N^2, \\ P_5(n) &= \tfrac{1}{12}N^2(2N-1), \\ P_7(n) &= \tfrac{1}{24}N^2\left(3N^2+4N-2\right), \\ P_9(n) &= \tfrac{1}{20}N^2(N-1)\left(2N^2-3N+3\right). \end{aligned}$$

***Table 3.** $P_1(n)$, $P_3(n)$, $P_5(n)$, ..., expressed in terms of $N = n(n+1)$.*

***Table 4.** $P_2(n)$, $P_4(n)$, $P_6(n)$, ..., expressed in terms of $N = n(n+1)$.*

$$
\begin{aligned}
P_2(n) &= \tfrac{1}{6}(2n+1)N, \\
P_4(n) &= \tfrac{1}{30}(2n+1)N(3N-1), \\
P_6(n) &= \tfrac{1}{42}(2n+1)N\left(3N^2-3N+1\right), \\
P_8(n) &= \tfrac{1}{90}(2n+1)N\left(5N^3-10N^2+9N-3\right), \\
P_{10}(n) &= \tfrac{1}{66}(2n+1)N\left(3N^4-10N^3+17N^2-15N+5\right).
\end{aligned}
$$

gave no hint of any kind of proof. His formulas go up to $k = 23$, and they have been found to be absolutely correct! The first formal proof of his formulas was given only in the nineteenth century, by Jacobi.

We shall now prove that these properties hold good for higher k, and we use the Bernoulli polynomials yet again for the proof. We already know that

$$P_k(x) = \frac{B_{k+1}(x+1) - B_{k+1}(1)}{k+1}.$$

We first show that if k is odd, then

$$P_k(-x-1) = P_k(x), \quad \text{for all } x.$$

For this, it suffices to show that $B_{k+1}(-x) = B_{k+1}(x+1)$, i.e., $B_{k+1}(-x) = B_{k+1}(1+x)$ for all x. This is the same as showing that $B_{k+1}(x) = B_{k+1}(1-x)$; but this follows from Property 3 (since $k+1$ is even).

So $P_k(-x-1) = P_k(x)$ for $k \geq 1$, odd. Since $x + (-x-1) = -1$, we express this by saying that if $k \geq 1$ and is odd, then $P_k(x)$ is symmetric about the point $x = -\frac{1}{2}$. This implies that $P_k(x)$ is a polynomial in $x+\frac{1}{2}$, and uses only *even* powers of $x+\frac{1}{2}$. Since $(x+\frac{1}{2})^2 = x(x+1)+\frac{1}{4}$, it follows that $P_k(x)$ is a polynomial in $x(x+1)$ for $k \geq 1$, odd.

Box 1. Mathematica Commands Needed for *Table* 1

Here are the *Mathematica* commands that were used by the author to display the polynomials $P_1(n)$, $P_2(n)$, $P_3(n)$, ..., in factorized form.

```
ClearAll[p];
SetAttributes[p, Listable];
p[n_, k_] := Factor[FullSimplify[Sum[i^k, {i, 1, t}]]] /. t -> n;
Do[Print[p[n, k]], {k, 1, 10}]
```

If k is even, then we have the relation

$$P_k(x) = \frac{B_{k+1}(x+1)}{k+1}, \quad \text{since} \quad B_{k+1}(1) = B_{k+1}(0) = 0.$$

Also, for even k we have: $B_{k+1}(1-x) = -B_k(x)$. So for even k we have:

$$P_k(-x) = P_k(x-1).$$

Since $(-x)+(x-1) = -1$, we express this by saying that for $k \geq 2$, even, $P_k(x)$ may be written using only *odd* powers of $x + \frac{1}{2}$.

In other words, for $k \geq 2$, even, $P_k(x)$ is equal to $2x+1$ times a polynomial in $x(x+1)$. This is just what we had set out to prove.

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Suggested Reading

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[3] *Bernoulli Polynomials*; entry in: http://en.wikipedia.org/wiki/Bernoulli_polynomials+

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