Simple Trigonometric Identities and Basic Calculus Leading to Interesting Series

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This article is mainly about using simple trigonometric identities and basic calculus to deduce some new and interesting series. In particular, I use basic aspects of Gamma and Beta integrations to deduce these series. These lead to interesting series for π or functions of π. One form of beta integration which is often used in this article is:

$$\int_0^{\pi/2} \sin^{2m-1}x\cos^{2n-1}x dx = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}. \qquad (1)$$

For any integral value of m, Γ(m+1) = m!. [1]

First I derive an alternate expression for $\sin(x)$ or $\cos(x)$ in terms of real, exponential function. This will be used in integrations involving logarithimic function of $\sin(x)$ or $\cos(x)$ as integrand to result in series. For any real x obeying $-1 \leq x \leq 1$ we have

$$ln(1-x) = -[x + x^2/2 + x^3/3 + x^4/4 \cdots] \qquad (2)$$

$$ln(1+x) = [x - x^2/2 + x^3/3 - x^4/4 \cdots] \qquad (3)$$

Adding (2) and (3) we get

$$ln(1-x^2) = -\sum_{n=1}^{\infty} x^{2n}/n \qquad (4)$$

$$(1-x^2) = e^{-\sum_{n=1}^{\infty} x^{2n}/n} \qquad (5)$$

As $\sin(x)$, $\cos(x)$ satisfy the criterion $0\leq |\sin(x)| \leq 1$, the above equations hold good for $\sin(x)$ or $\cos(x)$. Hence we get

$$\sin^2(x) = e^{-\sum_{n=1}^{\infty} \cos^{2n}(x)/n} \qquad (6)$$

$$\cos^2(x) = e^{-\sum_{n=1}^{\infty} \sin^{2n}(x)/n} \qquad (7)$$

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Equation (7) can be rewritten as, for example,

$$\cos^2(x) = exp(-\sum_{n=1}^{\infty}\sum_{r=0}^{n} {}^nC_r \frac{(-1)^r}{n} \cos^{2r}(x)), \qquad (8)$$

and hence

$$ln(\cos^2 x) = -\sum_{n=1}^{\infty}\sum_{r=0}^{n} {}^nC_r \frac{(-1)^r}{n} \cos^{2r}(x). \qquad (9)$$

Integrating both sides of equation (9) between 0 and $\pi/2$, we get

$$\int_0^{\pi/2} ln(\cos^2 x)dx = -\sum_{n=1}^{\infty}\sum_{r=0}^{n} {}^nC_r \frac{(-1)^r}{n} \int_0^{\pi/2} \cos^{2r}(x)dx \qquad (10)$$

The LHS is $-\pi$ ln 2 [1]. The integral on RHS of (10) can be evaluated using equation (1). After rearrangement equation (10) can be written as

$$ln4^{\pi} = \sum_{n=1}^{\infty}\sum_{r=0}^{n} \frac{\Gamma(r+\frac{1}{2})\Gamma(n+1)\Gamma(\frac{1}{2})(-1)^r}{n\Gamma^2(r+1)\Gamma(n-r+1)}, \qquad (11)$$

where we made use of the expression ${}^nC_r = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$. As $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, equation (11) can be rewritten as

$$ln4^{\sqrt{\pi}} = \sum_{n=1}^{\infty}\sum_{r=0}^{n} \frac{\Gamma(r+\frac{1}{2})\Gamma(n+1)(-1)^r}{n\Gamma^2(r+1)\Gamma(n-r+1)}. \qquad (12)$$

The series in RHS converges slowly; neverthless it is interesting from an academic point of view.

Let us try to use equations (6) and (7) as derived here, in another integration involving logarithm of sine or cosine function as integrand. For example it can be seen [1] that

$$\int_0^{\pi/2} (ln\ \sin\ x)^2\ dx = (ln^2 2 + \frac{\pi^2}{12})\frac{\pi}{2}. \qquad (13)$$

Substituting for sin(x) from equation (6), we get

$$\int_0^{\pi/2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\cos^{2(n_1+n_2)} x}{4n_1 n_2}\, dx = (ln^2 2 + \frac{\pi^2}{12})\frac{\pi}{2}. \quad (14)$$

Evaluating the above β integration occurring on LHS of equation (14) we get

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\Gamma(n_1 + n_2 + \frac{1}{2})}{4n_1 n_2 \Gamma(n_1 + n_2 + 1)} = (ln^2 2 + \frac{\pi^2}{12})\sqrt{\pi} \quad (15)$$

In this excercise we have illustrated a method to obtain new series such as those shown in (12) and (15). Summarising, in the first part of the article we have written $\sin(x)$ or $\cos(x)$ in terms of exponential of series, so that integration with the integrand as either logarithm or function of logarithm of $\sin(x)$ or $\cos(x)$ would yield simple Beta integrations. These can be solved to obtain new and interesting series.

In the following part of the article we will use simple trigonometric equations in several integrations reduceable to β integrations to obtain some interesting series. We shall start with a simple integration

$$\int_0^{\pi/2} (\sin^2 x + \cos^2 x)^m dx = \pi/2. \quad (16)$$

This can be written as

$$\sum_{r=0}^{m} {}^mC_r \int_0^{\pi/2} \sin^{2(m-r)} x \, \cos^{2r} x dx = \pi/2, \quad (17)$$

which simplifies to

$$\sum_{r=0}^{m} \frac{\Gamma(m - r + \frac{1}{2})\Gamma(r + \frac{1}{2})}{\Gamma(m - r + 1)\Gamma(r + 1)} = \pi. \quad (18)$$

As another example let us start with the β integration which can be rewritten as

$$\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \, (\sin^2 x + \cos^2 x)^p dx = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \quad (19)$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}x\cos^{2n-1}x \ \sum_{r=0}^{p} {}^pC_r \sin^{2p-2r}x\cos^{2r}x \ dx = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}, \tag{20}$$

$$\Rightarrow \sum_{r=0}^{p} \frac{\Gamma(p+1)}{\Gamma(r+1)\Gamma(p-r+1)} \times \int_0^{\pi/2} \sin^{2p+2m-2r-1}x\cos^{2n+2r-1}x \ dx = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \tag{21}$$

$$\Rightarrow \sum_{r=0}^{p} \frac{\Gamma(p+1)\Gamma(p+m-r)\Gamma(n+r)}{\Gamma(r+1)\Gamma(p-r+1)\Gamma(p+m+n)} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{22}$$

If m=1−n in equation (22), we get

$$\sum_{r=0}^{p} \frac{\Gamma(p+1-n-r)\Gamma(n+r)}{\Gamma(r+1)\Gamma(p-r+1)} = \frac{\pi}{\sin(n\pi)}, \tag{23}$$

since $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$. By substituting $n = 1/2$ in equation (23), we get

$$\sum_{r=0}^{p} \frac{\Gamma(p-r+\frac{1}{2})\Gamma(r+\frac{1}{2})}{\Gamma(r+1)\Gamma(p-r+1)} = \pi, \tag{24}$$

which is the same as equation (18).

In this article, we have illustrated with some simple examples how to get interesting series using basic aspects of calculus. I hope that the readers, mainly university students, will be able to make use of some of these basic concepts to get more new and exciting series.

Suggested Reading

[1] J S Gradshteyn and J M Ryzhik, ***Table of Integrals, Series and Products***, 6th edition, Edited by Alan Jeffrey, Academic Press Inc., UK (31 Dec 1993).