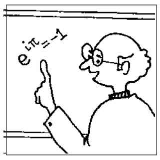
Classroom



In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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A Potpourri of Fermi Problems

A large number of problems can be solved, at least approximately, without recourse to detailed data and/or detailed calculation. Such problems are called Fermi problems. We shall give examples of Fermi problems which have been adapted from the 37th International Physics Olympiad (IPO) held in Singapore from 8–17 July, 2006. We believe that a study of these problems will encourage readers to develop the art of solving a wide range of problems using back of the envelope techniques.

1. Introduction

As a teacher at Indian Institute of Technology - Kanpur, one of us (VAS) had once posed the following question in a class: How many bicycle repair shops are there in the IIT – Kanpur campus? The cognoscenti would recognize this as yet another example of a Fermi problem (see box). The answer was to be provided on the spot and without recourse to any empirical survey. The students, considered to be among the brightest in the nation, were stumped. After some hesitation I encouraged them to answer the question step by step. The

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student population in campus was 3000 and we assume that 75% of them had bicycles. There were a 1000 resident families and two bicycles per family was a reasonable guess-estimate. Assuming that a bicycle would need some repair once a week we get that the number of bicycles visiting the friendly neighborhood repair person is 4250 per week. Given a six-day week the number of visits per day is 700. Assuming that each shop handles 60 bikes per day (say 10 minutes per bicycle), we get the number of cycle shops as 12. A survey revealed 11 shops. Not bad for an armchair theorist.

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The range of problems one can tackle without recourse to a handbook of detailed data is enormous. The exercise in Fermi problem goes by a variety of names. Some call it 'back of the envelope calculations'. There is a well known story about the Nobel laureate H G Bethe who arrived at the correct sequence of nuclear reactions determining the energy production in the sun and the stars, while traveling in a train. George Gamow described this as a great example of "back of the envelope calculation"[1]. Others call it order of magnitude calculations. Weisskopf described it as "the joys of creative laziness" [2]. We argue that the need for developing Fermi problem skills is greater with the advent of the Internet. Instead of getting buried under an avalanche of data and facts and spending hours sifting through them it is better to guess-estimate the data and forge (pun intended) the solution.

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2. Fermi Problems

In the IPO, 2006 one of the problems posed to over 400 of the brightest students across the globe consisted of a potpourri of Fermi-type problems [3]. Two of the problems presented below are inspired from this olympiad.

2.1 *Optics of the Digital Camera*

Consider a digital camera with a square charge couple

A digital camera's effectiveness as a photographic tool is determined by two effects: diffraction by the aperture and the pixel size.

device (CCD) chip with linear dimension $L = 35$ mm having $N_p = 5$ Mpix (1 Mpix = 10^6 pixels). The lens of this camera has a focal length of $f = 38$ mm. The well known sequence of numbers (2, 2.8, 4, 5.6, 8, 11, 16, 22) that appear on the lens refer to the so called f-number, which is denoted by F and defined as the ratio of the focal length to the diameter D of the aperture of the lens, $F = f/D$.

a) Find the best possible spatial resolution Δx_{min} at the chip limited by the lens. Express your result in terms of the wavelength λ and F.

Solution: The camera's effectiveness as a photographic tool is determined by two effects: diffraction by the aperture and the pixel size. For diffraction, the inherent angular resolution θ_R is given by the Rayleigh criterion,

$$\theta_R = 1.22\frac{\lambda}{D},$$

where we take λ to be the average wavelength of the incident light. A large pixel size results in image discontinuity while pronounced diffraction causes blurriness. When taking a picture, the object is generally sufficiently far away from the photographer for the image to form in the focal plane of the camera where the CCD chip should thus be placed. Thus

$$\Delta x_{\min} = f\theta_R = 1.22\lambda\, F. \tag{1}$$

To estimate $\Delta x_{\min}$ we take $\lambda = 500\,\text{nm}$, the mean wavelength of visible light. This yields $\Delta x_{\min} = 1.22\,\mu\text{m}$ (assuming the largest possible aperture or $F = 2$).

b) Find the necessary number N of Mpix that the CCD chip should possess in order to match this optimal resolution.

Solution: The number of pixels is given by a straightforward geometrical exercise

$$N = \left(\frac{L}{\Delta x_{\min}}\right)^2 = \left(\frac{35\,\text{mm}}{1.22\,\mu\text{m}}\right)^2 \approx 823\,\text{Mpix}.$$

Note that for a 5 Mpix camera the pixel size is l= 15.65 μm.

c) Photographers sometimes try to use a camera at the smallest practical aperture. Assume that we have a camera $N = 16$ Mpix with the chip size and focal length as given above. Which value is to be chosen for F such that the image quality is not limited by the optics?

Solution: Now looking for the unknown optimal aperture, we note that we should have pixel size $l \geq \Delta x_{\min}$ or in other words $F \leq F_0$. Using equation (1)

$$F_0 = \frac{L}{1.22\lambda\sqrt{N_0}} = 14.34.$$

Since this F value is not available, we choose the nearest value that has a higher optical resolution

$$F_0 = 11.$$

2.2 *The Hard-Boiled Egg*

An egg, taken directly from the refrigerator at temperature $T_0 = 4°$C, is dropped into a pot with water that is kept boiling at temperature T_1=100°C. Assume that the egg is spherical with radius r=2.5 cm and the coagulation temperature of albumen (egg protein) is $T_c = 65°$C. Also take the thermal conductivity $\kappa = 0.64$ WK^{-1} m^{-1} and assume it to be the same for liquid and solid albumen. Estimate the following:

a) The amount of energy U that is needed to get the egg coagulated.

Solution: The egg has to reach coagulation temperature. This means that the increase in temperature has

In a Fermi exercise, one is called upon to intelligently guess-estimate the unknown quantities.

to be

$$\Delta T = T_c - T_0 = 65^{\text{o}}\text{C} - 4^{\text{o}}\text{C} = 61^{\text{o}}\text{C}.$$

Thus the minimum amount of energy that we need to get into the egg such that all of it has coagulated is given by

$$U = \rho V C \Delta T,$$

where V is the volume of the egg and ρ is the density of the egg. In a Fermi exercise, one is called upon to intelligently guess-estimate the unknown quantities. It is well known that most land based animals have density slightly greater than that of water. The fact that we can swim is a testimonial to this. We take $\rho = 10^3$ kg-m^{-3}, the same as that of water. In the same fashion we will take specific heat capacity $C = 4.2 \times 10^3$ J-kg^{-1}, the same as that of water. We thus find

$$U = \rho \frac{4\pi r^3}{3} C(T_c - T_0) = 16768\,\text{J}.$$

b) The heat flow J into the egg.

Solution: From simplified heat flow equation, the total heat flowing into the egg through its surface is

$$J = \kappa(T_1 - \frac{T_c + T_0}{2})/r = 1676.8\,\text{W-m}^{-2}.$$

c) The power transfered to the egg.

Solution: This heat is transferred from the boiling water to the egg through the surface of the egg. So the heat power transferred to the egg is

$$P = 4\pi r^2 J \approx 13.1\,\text{W}$$

This is the amount of energy transferred to the egg per unit time.

d) The time needed to cook the egg so that it is hard-boiled.

Solution: From above estimation, time t required to transfer necessary amount of heat to egg

$$t = \frac{U}{P} = \frac{16768}{13.1} \approx 21.3 \text{ min.}$$

We wish to point out that there have been many labored attempts to obtain the cooking time of an egg. Elaborate formulae have been proposed elsewhere [4].

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2.3 *Capillary Vessels*

Let us regard blood as an incompressible viscous fluid with dynamic viscosity η= 4.5 g-m^{-1}-s^{-1}. We model a capillary as a circular straight pipe (cylindrical pipe) of radius $r = 4\,\mu$m , length $L = 1$ mm, and assume that all capillaries in the human body are connected in parallel.

For the systemic blood circulation (the one flowing from the left ventricle to the right auricle of the heart), the blood flow is $D = 100$ cm^3-s^{-1} for a man at rest. It is given that the pressure drop across the capillary Δp=1 kPa. Estimate the following:

a) Total number of capillary vessels in the human body.

Solution: We employ Poiseuille's law which is analogous to Ohm's law. Since the capillaries are in parallel

$$\frac{1}{R_{\text{all}}} = \frac{N}{R},$$

where R is the hydraulic resistance of one capillary. Hence the number of capillary vessels in the human body

$$N = \frac{R}{R_{\text{all}}}.$$

We now calculate R using Poiseuille's law,

$$R = \frac{8\eta L}{\pi r^4} \approx 4.5 \times 10^{16} \text{kg-m}^{-4}\text{-s}^{-1}.$$

Hence

$$N \approx \frac{4.5 \times 10^{16}}{10^7} = 4.5 \times 10^9.$$

We digress to point out that if we put all the capillaries of a single human being end to end, the total length $NL = 4.5 \times 10^9$mm $= 4500$ km!

b) Speed v with which blood is flowing through a capillary vessel.

Solution: The volume flow is $D = \pi r^2 v$. Hence, using "Ohm's" law for fluid flow (e.g. $RD = \Delta p$) we obtain

$$v = \frac{r^2 \Delta p}{8\eta L} = 0.44 \text{ mm-s}^{-1}.$$

2.4 *Frames in an Animation Movie*

'Hanuman' was one of the first full length fast moving Indian animation movie. Some of us must have watched it. Estimate the total number of frames in this 100 minute movie.

Solution: From our knowledge of the persistence of vision, the minimum number of frames per second (FPS) is 16–20 to ensure continuity. We also know that normal theatrical movies run at 24 FPS . Taking into account that the movie includes fast moving frames, we assume that on an average there are 30 frames per second in the above mentioned high quality movie. Since it is 100 minutes long, the total number of frames are $100 \times 60 \times 30 = 1{,}80{,}000$. Note that the exact figure is 2,00,000 which is closer to our Fermi type estimation [5].

The largest telescope in meter wave range is situated in India. It is called the Giant Meterwave Radio Telescope.

2.5 *GMRT*

The largest telescope in meter wave range is situated in India. Called the Giant Meterwave Radio Telescope (GMRT), it consists of 30 parabolic dishes spread out in Y shape in a vast area near Pune. Each dish has a size of 45 meter. The GMRT can observe 153 MHz radiation at 20 arc sec of resolution. Estimate the effective size of the setup (i.e. the size of the virtual telescope).

Solution: Once again we take recourse to the famous Rayleigh criterion which was introduced in Problem (2.1). The resolution can be approximated by

$$= 1.2\frac{\lambda}{D}.$$

Converting resolution in radians (10^{-4} rad) and frequency to wavelength ($\lambda = 2$ m) yields

$$D = 1.2 \times \frac{2}{10^{-4}} \approx 24 \text{ km}.$$

It shows that virtually we are using a radio dish of 24 km diameter. Note that actual setup is in 25 km diameter (see *Figure* 1)). On the other hand if we use a single dish of 45 meter for observing the same wavelength then it

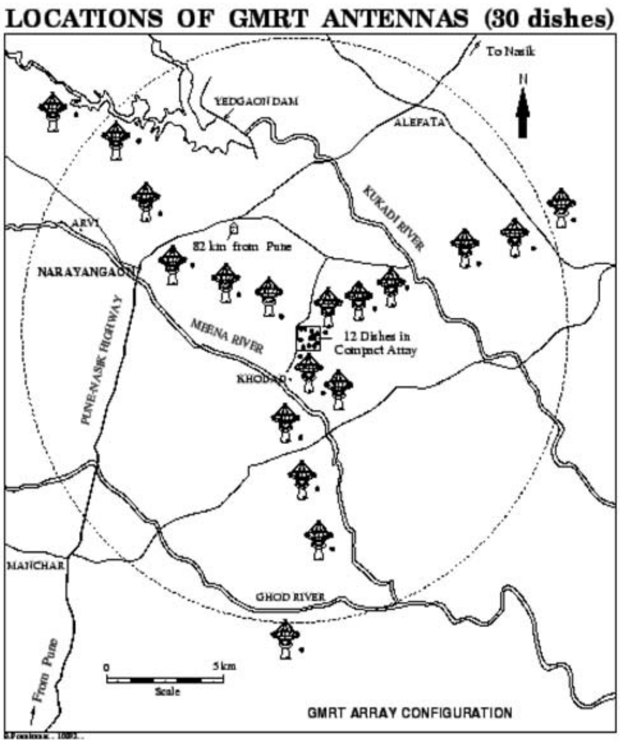

Figure 1. GMRT array configuration.

Box 1.

Enrico Fermi (1901-1954), a famous physicist, used to solve legendary estimation problems. Below is the summary solution of his classic Fermi problem "*How many piano tuners are there in Chicago*?"

a) There are approximately 3,000,000 people living in Chicago.
b) There may be an average of 4 people per household in Chicago, so the number of households is about 750,000.
c) Maybe about 1 in 10 households has a piano. This suggests that there are about 75000 pianos in Chicago.
d) Assuming a piano is tuned once a year, then 75000 piano tunings are needed. If a piano tuner tunes approximately 3 pianos a day, and works 200 days a year, the number of tuners needed is about 125.
e) So there are about 125 tuners in Chicago.

An empirical survey gives this result to be 130, which is close to the estimated result.

will yield a resolution of

$$1.2 \times \frac{2}{45} = 0.05\,\text{rad} = 3^0.$$

3. Conclusion

In fact there is an enormous range of problems that one can address using "Fermi tactics". In this venture your arsenal should comprise of scaling, dimensionality, knowledge of constants and formulae of elementary physics and the willingness to guess-estimate. Boldness, ingenuity, intuition, confidence, brashness, and the willingness to accept that you can be outrageously wrong are the hallmarks required. In other words, all the qualities of a good physicist (and a good human being).

Acknowledgment

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Suggested Reading

[1] George Gamow, *Birth and Death of the Sun*, Dover Publications, 2005.
[2] H Lin, *Fundamentals of Zoological Scaling*, *Am. J. Phys.*, Vol.50, pp.72–81, 1982.
[3] India participated in this prestigious event with a team of 5 students. All participants got medals including Raghu Mahajan (Chandigarh) – Gold, Mehul Tikekar (Mumbai) – Gold, Divyanshu Jha (Patna) - Bronze, Harish Ravi (Bangalore) – Bronze, and Neha Rambia (Mumbai) – Bronze.
[4] *New Scientist*, Vol.2138, p.105, 13 June 1998.
[5] http://www.audarya-fellowship.com/forums/hindu-sadhanas/178778-dont-miss-hanuman-animated-epic-movie.html