Borromean Triangles and Prime Knots in an Ancient Temple

Arul Lakshminarayan

The article is about mathematical motifs of modern interest such as remarkable links and knots that were found at a 6th century Chennai temple. Introducing these mathematical objects we discuss their timeless and universal appeal as well as point to possible reasons for their symbolic use in ancient Indian temples.

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1. Introduction

On a recent visit to an ancient temple in Thiruvanmayur, South Chennai, we observed that there are a series of pillars with beautiful and remarkable geometric designs that are mathematical motifs of contemporary scientific interest. The plausible reasons for their appearance in this temple are not hard to see, but we use this as an opportunity to relate a fascinating bit of story involving recurrent universal symbolism, physics and mathematics. The temple is named after the presiding deity 'Marundheeswarar', which translates to 'Lord of Medicine'. The motifs surround the sanctum of the goddess 'Tripurasundari', the "beautiful lady of the three cities". The number three is crucial in these motifs and the irreducible tripartite nature of the divinity is emphasized through the topology of links and knots.

The first of the patterns is a set of three identical overlapping equilateral triangles at whose center is a four petalled flower (*Figure* 1). Unlike the two-dimensional *yantras* which typically have several overlapping triangles, this one is sculpted with the third dimension in mind; we can make out when one triangle goes 'over' another. The three triangles overlap in a very specific and remarkable way: no two of the three triangles are linked to each other, but the three are inextricably collectively linked; if any one

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Figure 1. The Borromean triangles in the Marundheeswarar temple.

of the triangles is removed the other two fall apart as well. From a physicist's viewpoint, there are no 'two-body' correlations while there is an essential 'three-body' correlation.

2. Brunnian Links, Borromean Circles and Triangles

Modern mathematics classifies this object as a 'Brunnian link'. Formally a link is a collection of 'knots' that do not intersect each other, but may otherwise be linked, such as simply two interlinked rings. A 'knot' conforms to our conventional notion of a string looping around itself, but mathematicians prefer that the ends of the string be joined together. A single string that is not knotted at all, and is therefore called an 'unknot', is topologically a circle. A Brunnian link (after H Brunn, German mathematician who published his work on knot theory in the late nineteenth century) is a link such that if any one of the components is removed, the remaining ones become 'trivial' and fall apart into unlinked unknots. It is clear that the three triangles of the Tripurasundari temple form precisely such a link.

From a physicist's viewpoint, in the pattern of triangles in *Figure* 1, there are no 'two-body' correlations while there is an essential 'three-body' correlation.

However the best known and simplest example of a Brunnian link is the Borromean circles, three circles interlinked in such a manner that no two of them are linked but all three are simultaneously linked (*Figure* 2). The name derives from the medieval aristocratic Borromeo family from northern Italy who used this symbol extensively, including in their coat of arms. It is presumed that it signified the inseparable union of three powerful families at that time. The symbol however has been found in several other places and predates the medieval Italian family. It appears that a version of the Brunnian links with three triangles appears on 7th century Scandinavian rune stones where the god Odin is shown with these symbols called 'valknuts', meaning slain warriors' knots. The links found on the pillars of the Marundheeswarar temple appear to be a symmetrical version of this symbol. Borromean motifs have also been found in Japanese shrines and family emblems, and in medieval Christian iconography where it reconciles monotheism with the potential polytheism implied by the Trinity of the Father, the Son and the Holy Ghost.

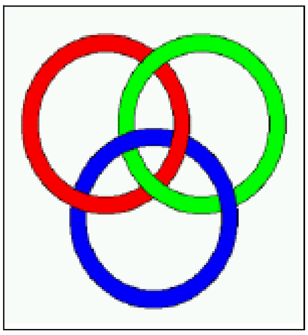

Figure 2. The classic Borromean rings.

While the Borromean circles and triangles are topologically similar, they are geometrically quite different. It has been established mathematically, fairly recently, that Borromean circles of any relative sizes are impossible. However it is possible to construct Borromean configurations with ellipses of arbitrarily small eccentricities . Thus there cannot be an actual three-dimensional realization of the Borromean circles. This constraint of geometry does not forbid Borromean configurations for other shapes such as ellipses, triangles or golden rectangles. Golden rectangles are those whose sides are of the golden ratio; three such orthogonal rectangles can be inscribed in a regular icosahedron (polyhedra with 20 faces, each an equilateral triangle). There are also Brunnian links when the number of components n is larger than 3. In this case there could be situations where no two components are linked, but there is a nontrivial sublink. Such links are called 'Borromean', as opposed to Brunnian where *every* single sublink is trivial.

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The Institute of Mathematical Sciences at Chennai, incidentally not far from the Marundheeswarar temple, has recently adopted this 'golden' version of the Borromean links for its logo. The

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International Mathematical Union adopted a three component Borromean link, which minimized the link length when tied with a rope of unit diameter, as their logo in August 2006. The Australian artist John Robinson has made Borromean sculptures using various shapes including triangles. It is indeed a lot of fun and challenge to take some cardboard and create ones own three-dimensional version of the Borromean triangles whose projection is found in the temple. Once created it is an object that is often the nucleus of conversations on Borromean matters! The universality of the Borromean symbol is very remarkable as it is found in disparate cultures and times. One of my personal favourites is a Borromean configuration involving a doughnut, a coffee mug and a computer mouse with cable which forms the logo of the Topological Quantum Computing project at Indiana University. This and other logos, as well as a considerable wealth of information on Borromean links is available in the website http://www.liv.ac.uk/~spmr02/rings, which is one of several sites that deal with matters Borromean. Ancient Indian uses of the symbol which must indeed be quite prevalent however do not seem to be documented.

Borromean Metaphors in Physics

The adjective Borromean is in use in few-body quantum systems – it describes the situation where a 2-body configuration is not stable, while a 3-body configuration may be. An example is provided by 'halo nuclei' with some neutrons loosely bound to a core, such as in the case of ^{6}He, which is stable against dissociation while ^{5}He is not. Thus while the 3-body configuration involving (α,n,n) is stable (there exists a bound state), the 2-body ones (α,n) and (n,n) involving a Helium nucleus (α) and a neutron (n) or just two bare neutrons are unstable (there are no bound states).

An example of a situation where a 2-body configuration is not stable, but a 3-body one is, is provided by 'halo nuclei' with some neutrons loosely bound to a core.

Another use of the Borromean links as a descriptive metaphor in quantum physics is in the study of entangled states. Entanglement is a peculiar quantum correlation that we do not observe in our everyday 'classical' world, but which could be a crucial resource for quantum computers, computers that are presumably much more powerful than the ones we use today. There are states of three particles such that any two of them are not entangled, that is, if one had access to only a pair of the particles they could be prepared in isolation, but if taken together the three are highly entangled and thus they cannot be prepared individually. Again as this is very suggestive of Borromean rings, physicist P K Aravind, of the Worcester Polytechnic Institute, talked of 'Borromean entanglement', although he used a slightly different notion for the equivalent action of removing one ring. The Greenberger–Horne–Zeilinger (GHZ) state

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$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\,(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle),$$

is of three spin 1/2 particles ($|\uparrow\rangle$ and $|\downarrow\rangle$ are states of spin-angular momentum of any one spin along the z direction being $\pm h/2$). Using the GHZ state, Aravind points out that measuring the z component of any one spin puts the other two in either the $|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$ state, in either case these are unentangled. The connections between knot theory and quantum entanglement are currently being studied.

There are many other contexts in which Borromean rings have appeared, for instance DNA and other molecules have been knotted into Borromean configurations recently. The French psychoanalyst Jacques Lacan used the Borromean links in his lectures, with the elements being the 'real', the 'symbolic' and the 'imaginary'. Apparently he considered that when any one of these aspects was absent it resulted in psychosis.

Some Speculations and a Prime Knot

Returning to the Thiruvanmayur temple, we could ask why such links were carved in their pillars. Although this is a forum for science education, I will venture to elaborate briefly. Indeed a moderate acquaintance with the 'esoteric' aspects of Hinduism gives us several plausible answers.

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Figure 3. A prime knot with six crossings, the Stevedore's knot, found in the Marundheeswarar temple.

As mentioned earlier, the pillars of interest are near the sanctum of the goddess *Tri-pura-sundari*. The recurrent theme of the triad is therefore to be expected. The archetypal mantra AUM contains three parts. The yogi's three principal nadis, the *ida*, the *pingala* and the central *sushumna* form a core tantric triad, which incidentally is also the symbolism of the staff of Caduceus with two snakes intertwinning around a staff. There are three *sakthis* or powers that derive from the goddess, the *iccha* (desire), the *gnana* (knowledge) and the *kriya* (action), and the symbol maybe emphasizing that without any one of these the other two are useless. There is of course the triad of Brahma, Vishnu and Shiva, or in a more esoteric sense there are three knots (*granthis*) in the human body, the *brahma granthi* in the lower body, the *vishnu granthi* in the region of the heart and the *rudra granthi* at the center of the eyebrows. Tripurasundari is the single goddess or power that devolves into these three knots that essentially creates, sustains and dissolves.

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This theme of a single power differentiating into three while remaining essentially one (the *Brahman* of the Upanishads) seems to be brought out in another symbol in a pillar of the same sanctum. This represents what looks like a snake knotting itself up into three parts, thus carrying within it again the irreducible triad. If the snake swallows its head or the ends are joined, we get the 'Stevedore's' knot, which is known from the mathematical theory of knots to be a 'prime knot' with six crossings. A prime knot cannot be composed of smaller simpler knots. Just as there are prime integers out of which all the whole numbers can be constructed, so also there are prime knots. Knots have been the object of mathematical and scientific interest since Lord Kelvin studied these in the latter half of the nineteenth century as a model of atoms (*vortex atoms*). This idea was soon abandoned, but the theory of knots stayed on and was developed into a beautiful subject. After a lull, there was a resurgence of interest in this subject from about twenty years ago, when seminal results were found and concrete connections to modern physics emerged.

Epilogue

Apart from the motifs noted above, the Tripurasundari sanctum pillars are decorated with other simpler geometric designs, notably the well-known Yin-Yang symbol, having a circle divided in two halves by smaller semicircles representing the Yin (female, *sakthi*) and the Yang (male, *siva*) energies. While the temple has been in existence from about the 6th century AD (it has been sung by Saivite saints of the 8th century) and it has at least 11th century inscriptions, I cannot comment on the era in which the pillars with the motifs discussed here were carved. Such motifs are also certainly not unique to this temple and the use of geometric patterns (*yantras*) is prevalent in both Hindusim and Buddhism. Further explorations may throw up more intriguing uses of mathematics to build bridges with the inner worlds that these temples seek to connect. The Borromean triangles or the Stevedore's knot as logos of Tripurasundari may be part of a larger spectrum.

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Suggested Reading

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[3] P K Aravind, Borromean entanglement of the GHZ state, in *Quantum Potentiality, Entanglement and Passion-at-a-Distance: Essays for Abner Shimony*, Ed. by R S Cohen, M Horne and J Stachel (Kluwer, Dordrecht, 1997) (Available at P K Aravind's homepage: http://users.wpi.edu/~paravind/publications.html)

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