Euclid's Fifth Postulate

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In this article, we review the fifth postulate of Euclid and trace its long and glorious history. That after two thousand years, it led to so much discussion and new ideas speaks for itself.

"It is hard to add to the fame and glory of Euclid, who managed to write an all-time bestseller, a classic book read and scrutinized for the last twenty three centuries." The book is called '*The Elements*' and consists of 13 books all devoted to various aspects of geometry and number theory. Of these, the most quoted is the one on the fundamentals of geometry Book I, which has 23 definitions, 5 postulates and 48 propositions. Of all the wealth of ideas in *The Elements*, the one that has claimed the greatest attention is the Fifth Postulate. For two thousand years, the Fifth postulate, also known as the 'parallel postulate', was suspected by mathematicians to be a theorem, which could be proved by using the first four postulates.

Starting from the commentary of Proclus, who taught at the Neoplatonic Academy in Athens in the fifth century some 700 years after Euclid to al Gauhary (9th century), to Omar Khayyam (11th century) to Saccheri (18th century), the fifth postulate was sought to be proved. Euclid himself had just stated the fifth postulate without trying to prove it. The main reason that such a proof was so much sought after, was that while other postulates appeared to be self-evident and obvious, the fifth postulate involving the intersection of lines at potentially infinite distances, was hardly self-evident. In these proofs it was unwittingly assumed that an equivalent axiom holds. This was invariably the one which was better known as Playfair's axiom, after the Scottish mathematician John Playfair (1748 – 1819). Besides, some like al-Tusi (13th century) and Girolamo Saccheri (1667 – 1733) tried to prove the postulate by a *reductio ad absurdum* method. In 1733, Saccheri, a professor of rhetoric, theology and philosophy

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at a Jesuit college in Milan, published a two volume work entitled *Euclid free of every flaw "Euclidus Vindicatus"*. The nineteenth century saw mathematicians exploring all possible alternatives to the fifth postulate and discovering logically consistent geometries. In the1820's Nikolai Lobachevsky and Janos Bolyai independently realized that entirely self consistent "non-Euclidean" geometries could be created in which the parallel postulate did not hold. It is probable that Carl Friedrich Gauss had actually studied the problem before that, but if he did so, he did not publish any of his results. The names of Bernhard Riemann (1826 – 1866) and Henri Poincaré (1854 – 1912) are associated with the development of resulting geometries. By the end of the last century, it was shown once and for all that the fifth postulate is *independent* of the remaining postulates and that all attempts at proving it using the other four were doomed from the beginning.

1. The Postulates

The five postulates have been cited over and over again, but a repetition will do no harm. They are:

1) A straight line can be drawn joining any two points.
2) Any straight line segment can be extended indefinitely in a straight line.
3) Given any straight line segment, a circle can be drawn having the segment as radius and an endpoint as centre.
4) All right angles are congruent.
5) If two lines are drawn, which intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must intersect each other on that side if extended far enough. (*Figure* 1.)

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The fifth postulate is also known as the Parallel Postulate, because it can be used to prove properties of parallel lines. Euclid used only the first four postulates for the first 28 propositions in Book I, but was forced to invoke the parallel postulate in proving Proposition 29. This states that a straight line falling on parallel straight lines makes the alternate angles equal to one another, the

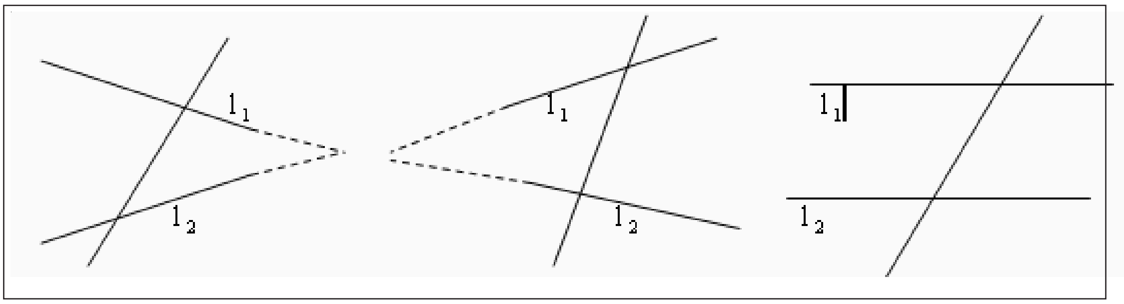

Figure 1.

exterior angles equal to the interior and opposite angle and the sum of the interior angles on the same side equal to two right angles.

Euclidean geometry is the study of geometry that satisfies all the five postulates of Euclid, including the parallel postulate.

2. Consequences of the Fifth Postulate

In a plane, given a line and a point not on the line, there are exactly three possibilities with regard to the number of lines through the point:

1) There is exactly one line parallel to the given line;
2) There is no line parallel to the given line;
3) There is more than one line parallel to the given line.

These are known as the hypotheses of the *right*, *obtuse* and *acute* angles. The first one is Playfair's axiom and is equivalent to Euclid's fifth postulate.

It is possible that Euclid chose not to use Playfair's axiom because it does not say how to construct this unique parallel line. With Euclid's original postulate, the construction of the parallel line is given as a proposition. The ancient Greeks declared objects "not to exist", if a construction could not be found for them.

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Consequences of the fifth postulate include:

1) If two parallel lines are cut by a transversal, then the corre-

sponding angles are equal, alternate angles are equal and interior angles are supplementary.

2) The sum of the angles of a triangle *equals* two right angles.

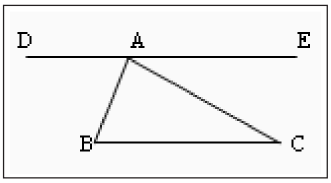

Figure 2.

The construction (*Figure* 2) requires one to draw a line DAE through the point A parallel to the opposite side BC. With that and using the fact that alternate angles are equal, one gets the sum of the angles of a triangle to be equal to 180^o.

3) If a line segment is drawn joining the midpoints of two sides of a triangle, it is parallel to the third side and equal to half its length.
4) The sum of the angles is the same for every triangle.
5) There exists a pair of similar, but not congruent, triangles.
6) If three angles of a quadrilateral are right angles, then the fourth is also a right angle.
7) Two lines, which are parallel to the same line, are parallel to each other.
8) Given two parallel lines, any line that intersects one of them also intersects the other.
9) In a right angled triangle, the square of the hypotenuse equals the sum of the squares on the other two sides.

3. Saccheri Quadrilaterals

Saccheri attempted to show that the assumption, that the parallel postulate of Euclid did not hold, would lead to a contradiction. Assuming that there is no parallel to a given line through a point not on it, he showed that it would contradict the second postulate of Euclid, which required straight lines to be infinitely long. Based on the fact that there was more than one parallel, he proved several counterintuitive statements, but could not formally obtain a logical contradiction. However, it is his use of quadrilaterals that is of interest.

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A quadrilateral, two of whose sides are perpendicular to a third side, is called a biperpendicular quadrilateral.

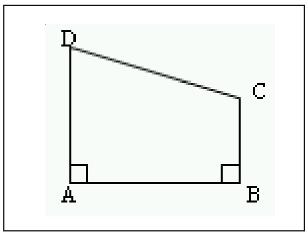


Figure 3.

AB is called the base and CD the summit. AD and BC are called the legs of the quadrilateral. ∠BCD and ∠CDA are called the summit angles. (*Figure* 3.)

Theorem 1. In a biperpendicular quadrilateral, the longer leg is opposite the larger summit angle.

This is true independent of the fifth postulate and depends only on the idea of a larger angle. If ∠BCD > ∠CDA, then AD > BC.

If the legs of the biperpendicular quadrilateral are of equal length, the quadrilateral is called a Saccheri quadrilateral.

Theorem 2. The summit angles of a Saccheri quadrilateral are equal. (Here, too, we do not use the fifth postulate or any of its equivalents). (*Figure* 4.)

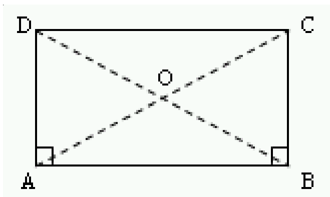


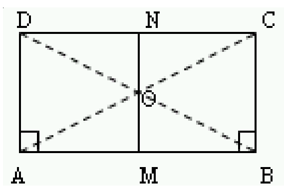
Figure 4.

Proof. Consider Δs DAB and CBA.
∠DAB = ∠CBA = 90°, AD = BC, AB is common.
By SAS postulate, Δs DAB and CBA are congruent.⇒ DB = CA.
Consider Δs ACD and BDC.
AD = BC, AC = BD, CD is common
By SSS postulate, Δs ACD and BDC are congruent.
⇒ ∠ADC = ∠BCD, i.e., the summit angles are equal.
Also ∠ACD = ∠BDC, so OD = OC and OB = OA.

Another theorem, which is required for further study, involves the line joining the midpoints of the summit and the base. This result is also true independent of whether the fifth postulate holds or not.

Theorem 3. The line segment joining the midpoints of the base and the summit is perpendicular to both of them. (*Figure* 5.)

Figure 5.



Proof. M is the midpoint of AB and N is the midpoint of CD. Join ON, OM.

By the previous theorem OD = OC. Also DN = CN and ON is common.

By SSS postulate, Δs OCN and ODN are congruent.
$\Rightarrow$ $\angle CNO = \angle DNO = 90^o$, i.e., ON is perpendicular to CD.
Similarly, since Δs OAM and OBM are congruent,
$\Rightarrow$ $\angle OMA = \angle OMB = 90^o$, i.e., OM is perpendicular to AB.
Besides, $\angle CON = \angle DON$, $\angle COB = \angle DOA$, $\angle BOM = \angle AOM$.
So $\angle CON + \angle COB + \angle BOM = \angle DON + \angle DOA + \angle AOM = 180^o$.
$\Rightarrow$ MON is a straight line. MN is perpendicular to both AB and CD.

Now that it has been shown that the summit angles of a Saccheri quadrilateral are equal, three possibilities arise:

1) They are both right angles.
2) They are both obtuse angles.
3) They are both acute angles.

Possibility 1. This follows immediately from the fifth postulate of Euclid. The proof follows from the fact that since the interior angles are supplementary, AD is parallel to BC. This together with the property that alternate angles are equal, leads to the fact that a Saccheri quadrilateral is a rectangle in Euclidean geometry. The proof is left to the reader.

Possibility 2. Suppose that the summit angles are equal and they are obtuse.
$\Rightarrow \angle BCD = \angle CDA$ and both are obtuse.
M, N, the midpoints of AB and CD, respectively, are such that the line MN is perpendicular to AB and CD. Consider the quadrilateral CNMB.
$\angle CNM = \angle NMB = 90^o$, so CNMB is a biperpendicular quadrilateral, where the summit angles $\angle MBC$ (right angle) $< \angle BCN$ (obtuse angle).
$\Rightarrow$ CN < MB by Theorem 1. Further M, N being midpoints of AB and CD, respectively, it follows that CD < AB.

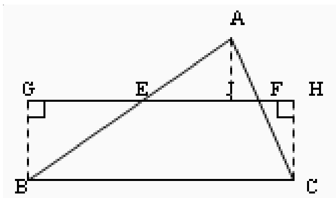


Figure 6.

This gives the following theorem:

Theorem 4. The summit of a Saccheri quadrilateral is shorter than its base.

Possibility 3. Suppose that the summit angles are equal and they are acute. Then, we have:

Theorem 4'. The summit of a Saccheri quadrilateral is longer than its base.

4. Further Consequences

There has been no contradiction so far. Let us examine further consequences of possibilities 2 and 3.

If Possibility 2 holds, i.e., the summit angles of a Saccheri quadrilateral are obtuse, then the following theorems follow:

Theorem 5. The sum of the angles of a triangle is greater than 180^o. (*Figure* 6.)

Proof . Let E, F be the midpoints of AB and AC, respectively. Join EF and from A, B and C drop perpendiculars AJ, BG and CH on EF.

In Δs BGE and AJE, $\angle BGE = \angle AJE = 90^o$, $\angle BEG = \angle AEJ$, BE = AE.

By ASA postulate, Δs BGE and AJE are congruent. $\Rightarrow$ BG = AJ, $\angle GBE = \angle EAJ$.

Similarly, Δs CHF and AJF are congruent. $\Rightarrow$ CH = AJ, $\angle HCF = \angle JAF$.

It follows that BG = AJ = CH. Also GE = EJ and JF = FH.

HGBC is a Saccheri quadrilateral with base GH and summit BC.

By Possibility 2, the summit angles of a Saccheri quadrilateral are obtuse, so it follows that

$\angle GBC + \angle HCB > 180^o$.

$\angle GBC = \angle GBE + \angle ABC = \angle EAJ + \angle ABC$.

$\angle HCB = \angle HCF + \angle ACB = \angle JAF + \angle ACB$

$\Rightarrow \angle GBC + \angle HCB = (\angle EAJ + \angle JAF) + \angle ABC + \angle ACB$.

$= \angle BAC + \angle ABC + \angle ACB > 180^o$.

The sum of the angles of a triangle is greater than 180^o.

Theorem 6 : The line segment joining the midpoints of two sides of a triangle is greater than half the third side.

Proof : In the Saccheri quadrilateral HGBC, summit BC < base GH.

$\Rightarrow$ BC < GE + EJ + JF + FH or

BC < 2 EJ + 2 JF or BC < 2(EJ + JF)

$\Rightarrow$ BC < 2 EF or EF > $\frac{1}{2}$ BC.

If instead, we had assumed Possibility 3, i.e., the summit angles of a Saccheri quadrilateral are acute, then Theorems 5 and 6 are modified as follows:

Theorem 5'. The sum of the angles of a triangle is less than 180^o.

Theorem 6'. The line segment joining the midpoints of two sides of a triangle is shorter than half the third side.

Unfortunately, Saccheri's work attracted little attention and was virtually unknown until over a hundred years later when it was republished by his compatriot Eugenio Beltrami. In 1766, Heinrich Lambert published a similar investigation. He too observed that results derived under Possibility 2 resemble those known for spherical geometry. He also suggested that the geometry assuming Possibility 3 might be visualized on a sphere of imaginary radius.

Lobachevsky and Bolyai built their geometries on Possibility 3. This is equivalent to the fact that given a straight line and a point not on it, more than one line could be drawn through the point parallel to the given line. This is also equivalent to Gauss' assumption that the sum of the angles of a triangle is less than 180^o. Possibility 2 was dismissed by Saccheri as contradicting the second postulate, namely, that every straight line can be extended

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indefinitely. Riemann was the first to notice that extending a line indefinitely did not require that the line be of infinite length. The geometry of Lobachevsky, Bolyai and Gauss is often called Hyperbolic Geometry because a line in the hyperbolic plane has two points at infinity, just as a hyperbola has two asymptotes. The geometry of Riemann is also called Elliptic Geometry, because a line in the plane described by this geometry has no point at infinity, where parallels may intersect it, just as an ellipse has no asymptotes.

5. Visualisation of Elliptic Geometry

Let V be the surface of a sphere in three-dimensional space. A great circle is a circle, which is the intersection of V with a plane through its centre. We can define a plane geometry by taking the plane to be the surface of the sphere and great circles to be lines on the plane. (*Figure* 7.) Riemann modified Euclid's Postulates 1, 2 and 5 to

Postulate 1'. Two distinct points determine at least one straight line. (Note that the end points of a diameter determine more than one line.)

Postulate 2'. A straight line is boundless, but not necessarily of infinite length.

Postulate 5'. Any two straight lines in a plane intersect.

All perpendiculars erected on the same side of a straight line are concurrent at its pole and are of the same length. (*Figure* 8.)

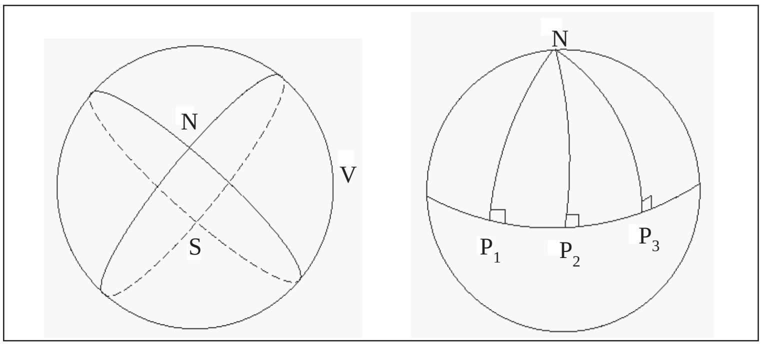


Figure 7 (left).
Figure 8 (right).

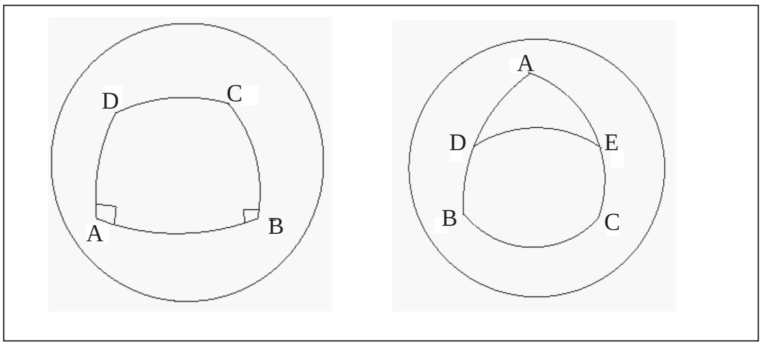


Figure 9.

Every straight line is of finite length equal to 4q, where q is the length of the perpendicular on that line from its pole. (*Figure* 8.)

In such a plane geometry, the sum of the angles of a triangle is always greater than 180^{o}; in a Saccheri quadrilateral, the summit angles are obtuse and the summit is shorter than the base; and the line joining the midpoints of two sides is greater than one half of the third side. (*Figure* 9.)

6. Visualisation of Hyperbolic Geometry

Consider a fixed circle C. The interior of C is taken to be the L-plane. An L-circle is a circle that is orthogonal to C. Define a geometry in the L-plane with L-circles and diameters as lines.

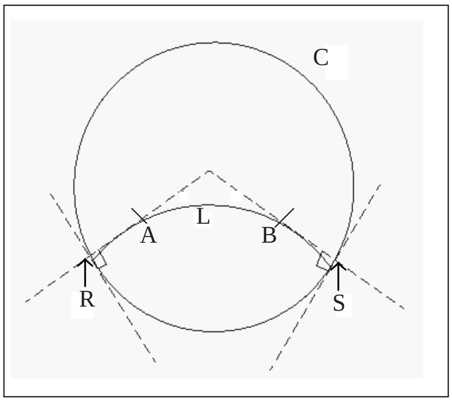
Postulate 1 of Euclid holds, namely that there is exactly one line through two distinct points.

Let A and B be two points on the L-plane. (*Figure* 10.) If R and S are the points at which the line through A and B intersects C, then the distance between A and B is defined as

$$d\,(A, B) = \left| \ln \frac{AR/AS}{BR/BS} \right| .$$

The distance postulates automatically hold. Postulate 2 of Euclid holds, namely, every line segment can be extended indefinitely.

Figure 10.



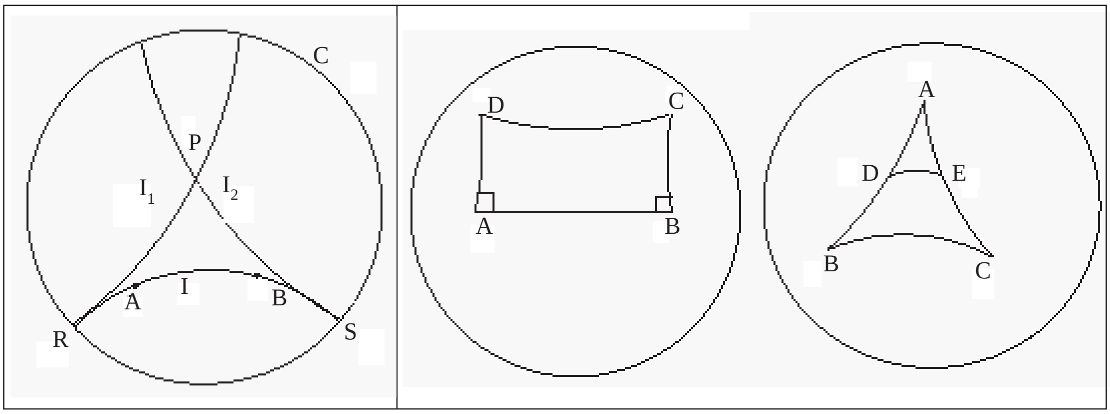


Figure 11 (left).
Figure 12 (right).

Postulate 5''. Given a line l and a point P not on it, there are two or more lines through the point P parallel to the given line l. (*Figure* 11.) Let the line l intersect C at R and S. The two bounding lines l_1 and l_2 are parallel to l.

In this geometry, the sum of the angles of a triangle is always less than 180º (*Figure* 11); in a Saccheri quadrilateral, the summit angles are acute and the summit is longer than the base; and the line joining the midpoints of two sides is less than one half of the third side (*Figure* 12).

The three geometries – that of (1) Euclid, (2) Riemann and (3) Lobachevsky, Bolyai and Gauss – form a triad and are a tribute to the ingenuity of humans, who have learnt not to accept anything without questioning. Mathematicians were forced to abandon the idea of a single correct geometry; it became their task not to discover mathematical systems but to create them by selecting consistent axioms and studying the theorems that could be derived from them.

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Suggested Reading

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