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On a Theorem of Vito Volterra

Students of (Real) Analysis who are introduced to the notion of continuity are fascinated by examples of continuous and non-continuous functions.. There are functions which are (i) continuous at exactly one point (or finitely many points) and nowhere else; (ii) continuous everywhere, but differentiable nowhere; (iii) continuous at irrationals but discontinuous at rationals.

But a striking result is that no real function can be continuous only at (and all) rationals. An elementary proof was given by Vito Volterra (1860–1940), an Italian mathematician who rose from utter destitution to become a principled mathematician.

Keywords
Continuity, discontinuity, rationals, irrationals, nested intervals, Baire category theorem.

We know that it is possible to construct a real valued function on $I = [0,1]$ which is discontinuous at every rational in I and continuous at every irrational in I. In fact, an example of such a function is as follows:

$$g(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ (p,q) = 1 \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Indeed, if a is any rational number in I, by the density of irrationals in $\mathbb{R}$, there is a sequence say $\{x_n\}$ of irrationals in I that converges to a. Then $g(x_n) = 0$, while $g(a) > 0$. Thus g is discontinuous at a. Now let b be any irrational number in I. For every $\epsilon > 0$, Archimedean property asserts the existence of a natural number N such that $\frac{1}{N} < \epsilon$. Note that there are only finitely many rational numbers in I with denominator less that N. Thus there exists a $\delta > 0$ such that for every rational number $\frac{p}{q} \in (b-\delta, b+\delta)$, we have $q > N$. This implies, for every $x \in (b-\delta, b+\delta)$, $g(x) < \epsilon$. Hence g is continuous at b.

Such an example is useful to illustrate the fact that there can be an $\mathbb{R}$-integrable function having a dense set of discontinuities. However, this example also raises a natural question: Does there exist a real function on I which is discontinuous at every irrational and continuous at every rational? From the point of view of students, there is no simple way, even to guess the answer. Usually, the nonexistence of such a function is established as a corollary of the Baire Category Theorem (normally excluded from the undergraduate syllabus). Though the Baire category theorem is very important (the proofs of open mapping theorem, uniform boundedness principle depend on this theorem), it turns out to be too sophisticated for an undergraduate student. Here we present a very simple proof of the nonexistence of such a function. The proof is given by the Italian mathematician Vito Volterra [1]. Interestingly, Volterra gave this proof in 1881, when he was a student, and 18 years before the Baire Category Theorem which appeared in 1899.

Does there exist a real function on *I* which is discontinuous at every irrational and continuous at every rational?

The following theorem (and many other historical episodes) can be found in the excellent recent book [2] on Functional Analysis.

Theorem: There is no real function on I which is continuous exactly at rational numbers.

Proof: Suppose that such a function f exists. Let x_0 be any rational number in $(0,1)$. By continuity of f at x_0, there is a $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subseteq (0,1) \text{ and } \mid f(x) - f(x_0) \mid < \frac{1}{2}$$

whenever $|x - x_0| < \delta$.

Choose a_1, b_1 such that $a_1 < b_1$ and $[a_1, b_1] \subseteq (x_0 - \delta, x_0 + \delta)$. Then

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{1}{2} + \frac{1}{2} = 1$$

There is no real function on *I* which is continuous exactly at rational numbers.

The proof uses density of rationals and irrationals and the nested interval theorem. These properties are direct consequences of the l.u.b.axiom for real numbers.

for all $x, y \in [a_1, b_1]$. Next we choose an irrational point, say x_1, in (a_1, b_1). The function g defined above is continuous at the irrational x_1, and by the same argument applied to x_1, there exist points c_1, d_1 such that $c_1 < d_1$ and $[c_1, d_1] \subseteq (a_1, b_1)$ and $|g(x) - g(y)| < 1$ for all $x, y \in [c_1, d_1]$. Thus for all $x, y \in [c_1, d_1]$, we have *both* $|f(x) - f(y)| < 1$ and $|g(x) - g(y)| < 1$.

Starting with the interval (c_1, d_1) in place of $(0, 1)$, we can repeat the above argument to get $c_2 < d_2$ and $[c_2, d_2] \subseteq (c_1, d_1)$ such that for all $x, y \in [c_2, d_2]$, we have both $|f(x) - f(y)| < \frac{1}{2}$ and $|g(x) - g(y)| < \frac{1}{2}$. Now repeat the argument to construct a nested sequence of closed intervals $\{[c_n, d_n]\}$ such that

$$(c_n, d_n) \supset [c_{n+1}, d_{n+1}]$$

and such that for all $x, y \in [c_n, d_n]$, we have both $|f(x) - f(y)| < \frac{1}{2^n}$ and $|g(x) - g(y)| < \frac{1}{2^n}$. By the nested interval theorem, $S = \bigcap_{n=1}^{\infty} [c_n, d_n] \neq \phi$. So let $z \in S$. We now show that f and g are both continuous at z. Let $\epsilon > 0$ be given. Let N be a natural number such that $\frac{1}{2^N} < \epsilon$. By the construction of the intervals $[c_n, d_n]$, we see that $z \in (c_N, d_N)$. So there exists a real number $\delta > 0$ such that $(z - \delta, z + \delta) \subset (c_N, d_N)$. Now for any $x \in (z - \delta, z + \delta)$, we have $x, z \in [c_N, d_N]$ so that both $|f(x) - f(z)| < \frac{1}{2^N} < \epsilon$ and $|g(x) - g(z)| < \frac{1}{2^N} < \epsilon$. Therefore, both f and g are continuous at z, leading to the contradiction that z is both rational and irrational. Thus there is no function on I which is continuous exactly at rationals.

This proof uses density of rationals and irrationals and the nested interval theorem. These properties are direct consequences of the l.u.b.axiom for real numbers. So this proof can be presented even in the first year calculus class. In fact, the power of the l.u.b. axiom can be emphasized by means of the above theorem.

Remark: From the above theorem we conclude that if a real function is continuous at all rationals, then it must also be continuous at some irrational. In fact, in [3] it is proved that such a function must be continuous on an uncountable dense set of irrational numbers.

Incidentally, Volterra was the first mathematician to point out a drawback of the Riemann integral by constructing an example of a differentiable function on $[a, b]$ having bounded derivative but the derivative is not $\mathcal{R}$–integrable; see [4]. Volterra is known to be a founder (along with Fredholm and Hilbert) of the theory of integral operators. His example of an operator whose spectrum is $\{0\}$ appears in almost every book on functional analysis for example, see [5]. He was a major contributor in the development of the theory of differential forms and surprisingly enough, a recent article [6] asserts that: what is today known as the Poincaré lemma is actually Volterra's theorem because he proved it much before Poincaré. In the later part of his life, he took active interest in mathematical biology and made substantial contributions in that area.

Volterra was the first mathematician to point out a drawback of the Riemann integral by constructing an example of a differentiable function on [*a*,*b*] having bounded derivative but the derivative is not $\mathcal{R}$–integrable;

Suggested Reading

[1] W Dunham, A historical gem from Vito Volterra, *Mathematics Magazine*, Vol. 63, pp.234–237, 1990.

[2] K Saxe, *Beginning Functional Analysis*, Springer-Verlag, New York, 2002.

[3] D Gauld, Did the young Volterra know about Cantor?, *Mathematics Magazine*, Vol. 66, pp.246–247, 1993.

[4] I Rana, *An Introduction to Measure and Integration*, Narosa, New Delhi, 1997.

[5] B V Limaye, *Functional Analysis*, Wiley Eastern Ltd., New Delhi, 1981.

[6] H Samelson, Differential forms, the early days; or the stories of Deahna's theorem and of Volterra's theorem, *American Mathematical Monthly*, Vol.108, pp.522–530, 2001.