which tends to $\theta^2/2$ as n tends to infinity. Thus the desired limit is $(1/2)(\cos^{-1} p_0)^2$.

For example, if $p_0 = 1/2$, the limit is $\pi^2/18$; and if $p_0 = -1/2$, it is $2\pi^2/9$. Note that p_0 may even assume the values $+1$ and -1, in which case the limits are 0 and $\pi^2/2$ respectively.

Solution to How Many Couples will Survive?

TIO question appeared in *Resonance*, Vol.11, No.8, p.103, 2006.

Problem: Suppose there are $2n$ individuals who constitute n married couples at a given initial date. We wish to consider the survivors at some given later date. Assume each individual having a probability p of surviving till the later date (and possibly more!) independent of the others. Let s denote the number of survivors at the later date and c denote the number of surviving couples (in which both partners are alive). The problem is to derive an expression for $E(c/s)$ and show that it does not depend on p. What about $E(s/c)$?

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Solution:

There can be several methods of solving this interesting problem. Two of them are described here.

First Method: Define an indicator random variable I_j as follows:

$$I_j = 1, \quad \text{if } j\text{-th couple survives among the } s \text{ survivors}$$
$$\quad = 0, \quad \text{otherwise}$$

where $j = 1, 2, \quad n$.

Clearly $E(I_j) = 0 \cdot P(I_j = 0) + 1 \cdot P(I_j = 1) = P(I_j = 1)$.

Evidently,

$$E(c/s) = E\left(\sum_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} E(I_j) = nP(I_j = 1) \qquad (1)$$

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because the chance of survival till the later date is the same for every couple!

Next, s survivors out of $2n$ individuals can be chosen in ${}^{2n}C_s$ ways. If the j-th couple has to survive among the s survivors, fixing the j-th couple leaves $2n-2$ individuals from whom remaining $s-2$ survivors can be chosen in ${}^{2n-2}C_{s-2}$ ways. As all outcomes are equally likely, we have

$$\begin{aligned} P(I_j = 1) &= \frac{{}^{2n-2}C_{s-2}}{{}^{2n}C_s} \\ &= \frac{s(s-1)}{2n(2n-1)}, \end{aligned}$$

after simplification.

Substituting this result in (1) gives

$$E(c/s) = \frac{s(s-1)}{2(2n-1)}$$

which is clearly independent of p.

Second Method: Define two indicator random variables as follows:

$$\begin{aligned} \text{Let } X_i &= 1, \text{if male of } i\text{-th couple survives among } s \text{ survivors} \\ &= 0, \text{otherwise} \\ \text{and } Y_i &= 1, \text{if female of } i\text{-th couple survives among the } s \text{ survivors} \\ &= 0, \text{otherwise} \end{aligned}$$

where $i = 1, 2, \ldots, n$.

Then

$$E(c/s) = E\left(\sum_{i=1}^{n} X_i Y_i\right) \qquad \text{(why?)} \qquad (2)$$

Now

$$P(X_i = 1) = \frac{s}{2n}$$

and

$$P(Y_i = 1/X_i = 1) = \frac{s-1}{2n-1}.$$

Since the chance of survival till the later date is the same for every couple, we have from (2),

$$\begin{aligned} E(c/s) &= nP(X_i = 1, Y_i = 1) \\ &= nP(Y_i = 1/X_i = 1)P(X_i = 1) \end{aligned}$$

(by multiplication theorem of probability)

$$\begin{aligned} &= n\left(\frac{s-1}{2n-1}\right)\ \frac{s}{2n} \\ &= \frac{s(s-1)}{2(2n-1)},\ \text{as desired.} \end{aligned}$$

However, it is of interest to note that the expression for $E(s/c)$ contains p. To see why, observe that if there are c couples at the later date, we already have $2c$ individuals surviving. Hence $E(s/c) = 2c+$ expected number of survivors among the remaining $(2n - 2c)$ individuals. Since each individual has a probability p of surviving till the later date independent of others, we immediately have

$$E(s/c) = 2c + (2n - 2c)p = 2np + 2c(1 - p),$$

an expression that does contain p.

Remark:

This problem was discussed by Daniel Bernoulli in the year 1768.