This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

TIO question appeared in *Resonance*, Vol.11, No.8, p.103, 2006.

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Keyword
Solution.

Solution to On an Intriguing Limit Problem

Problem: Here is an intriguing limit problem. Let $-1 < p_0 < 1$ and define recursively

$$p_n = \left(\frac{1 + p_{n-1}}{2}\right)^{1/2}, \; n \in N.$$

Further define $P_n = 4^n(1-p_n)$. What happens to P_n as $n \to \infty$?

Observe that any attempt to express p_n in terms of p_0 leads to complicated expressions and there seems no way to simplify these expressions so that the limit may be conveniently taken. How do we proceed?

Solution:

As $-1 < p_0 < 1$, there is a θ such that $0 < \theta < \pi$ and $p_0 = \cos\theta$. Since $1 + \cos x = 2\cos^2(x/2)$, we get successively, $p_1 = \cos(\theta/2)$, $p_2 = \cos(\theta/2^2)$, $p_n = \cos(\theta/2^n)$, Note that although p_0 may be negative we have $p_1 > 0$, $p_2 > 0$, $p_n > 0$, Hence

$$P_n = 4^n(1-\cos\frac{\theta}{2^n}) = 4^n \cdot 2 \cdot \sin^2\frac{\theta}{2^{n+1}} = \frac{\theta^2}{2}\left(\frac{\sin\frac{\theta}{2^{n+1}}}{\frac{\theta}{2^{n+1}}}\right)^2$$

which tends to $\theta^2/2$ as n tends to infinity. Thus the desired limit is $(1/2)(\cos^{-1} p_0)^2$.

For example, if $p_0 = 1/2$, the limit is $\pi^2/18$; and if $p_0 = -1/2$, it is $2\pi^2/9$. Note that p_0 may even assume the values $+1$ and -1, in which case the limits are 0 and $\pi^2/2$ respectively.

Solution to
How Many Couples will Survive?

TIO question appeared in *Resonance*, Vol.11, No.8, p.103, 2006.

Problem: Suppose there are $2n$ individuals who constitute n married couples at a given initial date. We wish to consider the survivors at some given later date. Assume each individual having a probability p of surviving till the later date (and possibly more!) independent of the others. Let s denote the number of survivors at the later date and c denote the number of surviving couples (in which both partners are alive). The problem is to derive an expression for $E(c/s)$ and show that it does not depend on p. What about $E\,(s/c)$?

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Solution:

There can be several methods of solving this interesting problem. Two of them are described here.

First Method: Define an indicator random variable I_j as follows:

$$I_j = 1, \quad \text{if } j\text{-th couple survives among the } s \text{ survivors}$$
$$= 0, \quad \text{otherwise}$$

where $j = 1, 2, \quad n$.

Clearly $E(I_j) = 0 \cdot P(I_j = 0) + 1 \cdot P(I_j = 1) = P(I_j = 1)$.

Evidently,

$$E(c/s) = E\left(\sum_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} E(I_j) = nP(I_j = 1) \qquad (1)$$

Keyword
Solution.