A View of Newton as a Mathematician

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In this article, we present certain topics such as quadrature of planar regions, introduced prior to the invention of calculus by Newton, and consider his other mathematical contributions, to binomial theorem and infinite series, cubics, theory of equations and imaginary roots. As the development of calculus and its enormous applications are well known, we will not discuss much about it.

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Isaac Newton (1642–1727) is introduced to us at school as the greatest physicist of all time who influenced the scientific attitude in Europe. But, later at college, we realize that Newton is more than a physicist. He was a mathematical genius (and recognized as one only after he joined Trinity College, Cambridge in the early 1660's). He understood Nature better than others, a rare ability especially among mathematicians. Thus he is regarded as a physicist and a mathematician of the highest order.

The development of calculus and its relationship to science and engineering is well known to the generation of the present day. The invention of calculus is attributed to Newton's curiosity in understanding the dynamics of Nature. He coined the terms *fluent* (a flowing quantity) for a variable and *fluxions* for the derivatives.

The notation dy/dx for derivative is due to Leibniz (1646–1716) who had also invented calculus independently more or less at the same period.

It is to be noted that most of the original work of Newton was done during the 20-year period from 1665 to 1685, although the work was published later.

Keywords
Fluent, fluxions, binomial theorem, quadrature, rectification, infinitesimal calculus, indivisibles, tangent problem, cubics, imagianry roots.

Binomial Theorem and Infinite Series

It is now established beyond doubt that the first major

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mathematical discovery of Newton dealt with the *binomial theorem* for a positive integral index and its generalization to other powers leading to *infinite series.* Let us see quickly how Newton arrived at the expansion. The coefficients of $(a+b)^n$ can be written in the form of *Pascal's triangle.* Slightly modified and rewritten as an array, it reads as follows:

$$\begin{array}{rcccc} n=0; & 1 & 0 & 0 & 0 \\ n=1; & 1 & 1 & 0 & 0 \\ n=2; & 1 & 2 & 1 & 0 \end{array}$$

The zeros on the right indicate that the expression is finite. The jth entry in the row $n=k$ can be obtained by adding the jth and $(j-1)$th entries in the row $n=k-1$, for $k = 1, 2, \cdot$. Newton used this observation to construct the entries of the row $n=-1$ starting the first element as 1 and constructing successively the elements by applying the same rule so that the row $n=-1$ produces the row $n=0$. He arrived at the entries $1, -1, 1, -1,$ for $n=-1$. He then extended it backwards to get the entries for $n=-2, -3,$ one by one as:

$$\begin{array}{rcccc} n=-3; & 1 & -3 & 6 & -10 \\ n=-2; & 1 & -2 & 3 & -4 \\ n=-1; & 1 & -1 & 1 & -1 \\ n=0; & 1 & 0 & 0 & 0 \\ n=1; & 1 & 1 & 0 & 0 \\ n=2; & 1 & 2 & 1 & 0 \end{array}$$

For the negative integers, the expression turned out to be infinite and he was indeed aware of the difficulties involved in dealing with the infinite series as its understanding was still at the premature stage. The expansion for fractional powers was obtained by the method of interpolation. By considering the expansion of $1 = (1-x^2)^{0/2}$, $(1-x^2) = (1-x^2)^{2/2}$, $(1-x^2)^2 = (1-x^2)^{4/2}$, he deduced a rule connecting the successive coefficients in the expression of $(1-x^2)^{1/2}$, $(1-x^2)^{3/2}$, and by analogy obtained the expression for the general case which can be stated in the following form:

$$(a+ab)^{m/n} = A + \frac{m}{n}Ab + \frac{m-n}{2n}Bb + \frac{m-2n}{3n}Cb +$$

where the first term is $A = a^{m/n}$, the second term is $B = \frac{m}{n}Ab$, the third term is $C = \frac{m-n}{2n}Bb$, and so on.

The infinite series was used extensively in the study of *quadrature* (finding the area of planar region bounded by closed curves by the method of squaring) and *rectification of curves.* Initially, in the case where the ordinate y is given as a function of x explicitly, for example, $y = \frac{a^2}{b+x}$, $y = (a^2 \pm x^2)^{1/2}$, $y = (\frac{1+ax^2}{1-bx^2})^{1/2}$, the quadrature was obtained by expanding the right hand sides as infinite series. He then derives quadrature of curves whose ordinate is in implicit form. Newton did not prove *convergence* in the modern mathematical sense which was done by Cauchy (1789–1857) and others after more than 100 years of endeavor.

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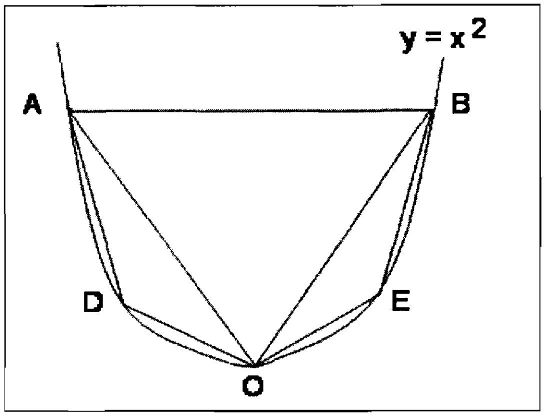
Why the Greeks did not Invent Calculus

In Newton's own words, *"If I had seen a little farther than others, it is because I have stood on the shoulders of giants"* Let us try to see who some of these giants may have been.

The central idea behind calculus better known as *infinitesimal calculus* is to use the *limit process* to derive results, and this goes back to the Greeks. To them the age-old concept of finding area of planar shapes and volume of solids was important. Archimedes[2] of Syracuse (287–212 BC) was involved in finding these quantities, particularly, for *conic sections.* He applied the *method of exhaustion* successfully to circles and parabolae. The method of exhaustion is actually attributed to Exodus (370 BC). Archimedes never used the term 'limit' as it

[2] See *Resonance*, Vol.11, No.10, 2006.

Figure 1.



was taboo to the Greeks. His idea was to inscribe and escribe a circle by a series of *regular polygons* of more and more sides whose perimeter can be calculated. In this way, Archimedes gave lower and upper bounds for π to an accuracy which was sufficient for most practical purposes.

The above method was indeed a milestone in mathematics and Archimedes was interested in other shapes pertaining to conical objects: parabolae, ellipses and hyperbolae. He started with the parabola (*Figure* 1) as it had a lot of applications such as the ability of a parabolic mirror to reflect light coming from infinity and concentrate it at a point, namely the *focus* (which means *fireplace*). He divided the parabolic sector (see *Figure* 1) into a series of triangles whose areas decrease in geometric progression and eventually 'exhaust' the entire region in the limit(!). He found that the total area 'approaches' $\frac{4}{3}area(ABO)$ (see *Figure* 1). The Greeks were never comfortable with infinity and Archimedes never used it in his arguments. Concerning other conic sectors, he failed in his attempts to find the areas of elliptical and hyperbolic sectors and it had to wait for two thousand years until the invention of *integral calculus.*

The method of exhaustion is close to modern integral calculus.

The method of exhaustion is close to modern *integral calculus.* The Greeks' failure can be attributed to at least two reasons; (i) their uneasiness with the concept of infinity as discussed earlier, and (ii) the fact that they did not know sufficient algebra. Indeed, the Greeks were masters of geometry, however their knowledge of algebra was minimal. As we know, algebra is a collection of *symbols* with a set of rules to work with them. The Greeks' failure in algebra was due to a lack of proper notation and their inability in dealing with equations. They really feared and avoided infinity as they suspected that wrong conclusions could easily be drawn which was amply demonstrated by Zeno's paradox that *motion is impossible.*

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Kepler and the Method of Indivisibles

The scientific activity in all spheres got rejuvenated around the fifteenth century after a long gap of more than 1500 years. Some of the prominent investigators were Nicolas Copernicus (1473–1543), Rene Descartes (1596–1650), Galileo Galilei (1564–1642), Johannes Kepler (1571–1630), Peirre de Fermat (1601–1665) and Blaise Pascal (1623–1662).

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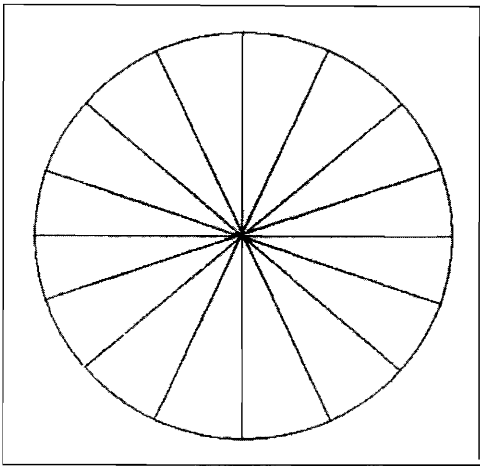
Kepler was the perfect representative of the period of transition from the old world to the new. His discovery of the three planetary laws which renewed the interest of scientists once again in the conic sections and the calculation of the area and rectification of curves. Kepler showed interest in Archimedes' method of exhaustion, but without much fear of infinity. By that time, the use of infinite products and sums was already in the literature especially in the context of π being represented as:

Francois Viete (1540–1603): $\frac{2}{\pi} = \frac{\sqrt{2}}{2} . \frac{\sqrt{2+\sqrt{2}}}{2}.$

John Wallis (1616–1703): $\frac{\pi}{2} = \frac{2}{1} \; \frac{2}{3} \; \frac{4}{3} \; \frac{4}{5} \; \frac{6}{5} \; \frac{6}{7}$

John Gregory (1638–1675): $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$



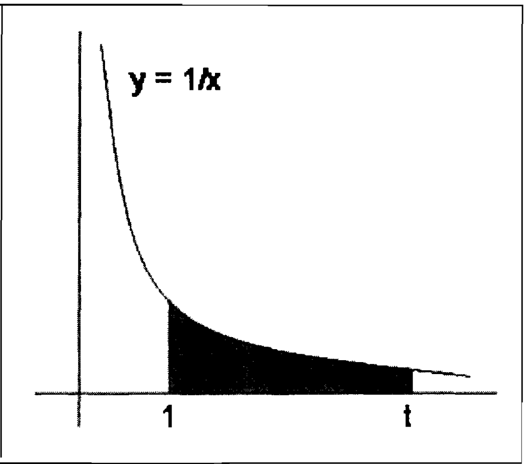


Figure 2 (left).
Figure 3 (right).

The method of indivisibles regards a planar shape as being made up of *a large number* of *infinitely small* strips, the so-called *indivisibles.* Though most of them did not understand what an indivisible was, the method seemed to work in many examples. A circle can be viewed as a collection of 'infinitely many' (in modern language, finitely many, but large) triangles with bases as infinitesimally small arcs (see *Figure* 2). Since the area of each triangle is half base times the radius, one arrives at the formula $A = Cr/2$, where A is the area and C is the circumference. This, of course, has been known since ancient times. For all its mathematical flaws, Kepler was the one to use the method of indivisibles to full potential to get new results.

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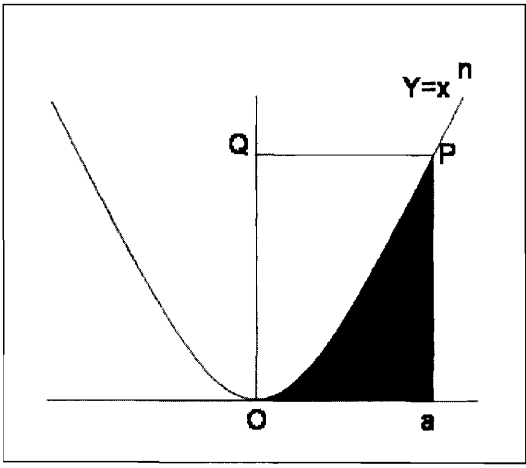
Squaring a hyperbola stubbornly resisted all attempts by many including Kepler. Consider the hyperbola $y = \frac{1}{x}$. By squaring it, we mean to find the area below the curve and above the x-axis and between two parallel lines $x = a$ (we may take $a = 1$) and $x = t$ (see *Figure* 3)

Fermat, Saint–Vincent and Quadrature of Hyperbola

The quadrature of generalized parabolae $y = x^n$, $n \in \mathbf{N}$, (*Figure* 4) came to the attention of Pierre de Fermat. His



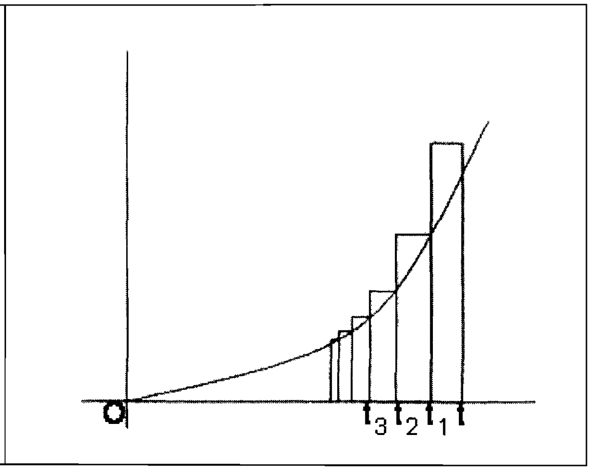


Figure 4 (left).
Figure 5 (right).

idea was to view the interval $(0, a)$ as an infinite number of intervals $(0, t)$, $(0, t_1)$, $(0, t_2)$, and so on (see *Figure* 5) in such a way that the lengths $t = a$, $t_1 = ar$, $t_2 = ar^2$, $\cdot$, $r < 1$ are in a geometric progression. Then evaluate the area of each strip viewing it as a rectangle to get the approximate area $A_r = \sum(ar^k)^n(ar^k - ar^{k+1}) = \sum(ar^k)^{n+1}(1-r) = \frac{a^{n+1}(1-r)}{1-r^{n+1}}$. Using $(1-r)(1+r+$ $+ r^n) = 1 - r^{n+1}$ and finally taking limit as $r \to 1$ (infinitely many indivisibles), we arrive at the formula $A = \frac{a^{n+1}}{n+1}$.

This work was significant on two grounds; (i) it works for a family of curves and (ii) the method extends to negative integers as well, but with an exception, namely, $n = -1$, which corresponds to the hyperbola $y = \frac{1}{x}$. The exceptional case was resolved to some extent by Saint-Vincent (1584–1667), who was not very famous, but was a contemporary of Fermat. A crucial observation in this case is that each strip produces an equal area, namely $\frac{ar^k - ar^{k+1}}{ar^k} = 1 - r$. This means that if the distances $t, t_1, t_2,$ from the origin are in geometric progression, then the corresponding areas reduce by a fixed quantity, i.e., the area is in arithmetic progression and this remains true as r becomes closer and closer to 1. This indeed shows that the relation between area $A(t)$ from 1 to t and distance t is *logarithmic.* That is $A(t) = \log t$;

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this was probably the first appearance of the use of logarithmic function which until then was mainly used for computational purposes. The quadrature of hyperbola is achieved, but with a hitch that the *base* of the logarithm is still unclear. One cannot choose an arbitrary base as in the case of a circle whose area is kr^2, where the constant k equals π and not any arbitrary number.

Newton, Leibniz and Calculus

Though Archimedes' method of exhaustion and the modified method of indivisibles solved the problem of quadrature for various curves, it was not on firm mathematical foundation. Moreover, each problem required a different approach which depended on the geometry (this is not surprising!) of the problem and algebraic skills. One needed to have a general approach with systematic procedure (an algorithm). This was achieved by the invention of differential and integral calculus by Newton and Leibniz, and it gained popularity in a short time in every field of science and engineering.

As is well known, differential calculus invented by Newton emerged from the study of *rate of change* of varying physical quantities. His dynamic view really goes back to antiquity and it reached its peak when Galileo, the father of experimental science, laid the foundation of mechanics prior to Newton.

The two fundamental problems, viz. the tangent problem and area problem are *inverse problems*, the crux of differential and integral calculus.

The problem of finding derivatives (this name came at a later stage) is known as the *tangent problem.* After the tangent problem, Newton concentrated on the *inverse* of the tangent problem, namely *antiderivative* or *integration.* Indeed by studying the inverse problem for curves like $y = x^n$, he arrived at the familiar expression $\frac{a^{n+1}}{n+1}$ earlier obtained by Fermat in the calculation of areas. Thus it is recognized that the two fundamental problems, viz. the tangent problem and area problem are *inverse problems*, the crux of differential and integral calculus. The relation connecting the two processes of finding the area and the derivative is known as the *Fundamental theorem of Calculus.*

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Newton did not give a proof (in the strict sense of rigour today) of it though he grasped its essence and full potential. The actual proof is due to Cauchy who laid the foundations of modern mathematical analysis more than a hundred years later. The invention of calculus affected every branch of science, more so in mathematics and was probably the single most remarkable finding after Euclid's classical geometry, nearly two thousand years earlier.

Calculus and Quadrature of the Hyperbola

The applicability of calculus need not be stressed here, but we wish to see how the quadrature of the hyperbola was achieved. Due to Saint–Vincent, it was reduced to finding the base of the logarithm. Among the calculation of derivatives, needless to say the *exponential function* $f(x) = b^x$, $b > 0$ got special attention and $f'(x)$ is proportional to $f(x)$, that is $f'(x) = kf(x) = kb^x$, where $k = \lim_{h\to 0} \frac{b^h-1}{h}$. In particular, the derivative is unchanged when we choose the base b such that $k = 1$. A heuristic derivation leads to $b = \lim_{m\to\infty}(1+\frac{1}{m})^m$, the *natural base* e. This curious number was probably recognized by the merchants or money lenders much earlier.

With this the problem of quadrature can easily be completed. Let $y = e^x$, or equivalently, $x = \log_e y = ln\ y$. We see from $\frac{dy}{dx} = y$ that $\frac{dx}{dy} = \frac{1}{y}$; that is $\frac{d}{dy}(\log_e y) = \frac{1}{y}$. In other words $\log_e y$ is the antiderivative of $\frac{1}{y}$. Now since the antiderivative is nothing but the area, it follows that $\log_e y = \int \frac{1}{y}dy$. This proves that e is the required base, justifying the name *natural base* which baffled this author from early days as to why e was used as the natural base rather than 10, called the *common base*.

A Tail to the Tale

The curious number e appears in connection with the hyperbola somewhat in a similar fashion as π does with the circle. The most fascinating fact is that these two numbers are connected by a third object i, the complex square root of -1 via one of the most beautiful equations in mathematics, namely $e^{i\pi} + 1 = 0$.

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Newton's contribution to other mathematical areas is also quite significant, and a few of them are indicated below.

Cubics

Apart from other results in cubics, Newton showed that many of the important properties of conics have their analogies in the theory of cubics. Further to the fundamental classification of the equations as *algebraic* and *transcendental*, by analyzing and studying the asymptotes, he classified the equations into four *characteristic forms*;

1) $xy^2 + hy = ax^3 + bx^2 + cx + d$,
2) $xy = ax^3 + bx^2 + cx + d$,
3) $y^2 = ax^3 + bx^2 + cx + d$,
4) $y = ax^3 + bx^2 + cx + d$.

Each of the above equations is further discussed in detail and there are 78 forms which a cubic may take. He enumerated 72, and the remaining ones were later mentioned by Stirling, Nicole and Nicolas Bernoulli.

The shadow given by equation (3) which is $y^2 = ax^3 + bx^2 + cx + d$ will give rise to all cubics.

It is well known that the shadow of a circle cast on a plane by an illuminating point gives rise to all the conics depending on the angle of illumination. Newton made the *remarkable observation* that the shadow given by the 3rd equation in the above list; $y^2 = ax^3 + bx^2 + cx + d$ will give rise to all cubics. Later this fact was demonstrated by Nicole and Clairaut and again by Murdoch in 1740 using certain classification of curves into five species based on the point of intersection with the x-axis.

Theory of Equations and Imaginary Roots

Among the many works of Newton on which he used to lecture during the years 1673 to 1683, the most interesting one is the extension of Descartes' rule of signs to give bounds to the number of imaginary roots. He used the principle of continuity to explain how two real and unequal roots may become imaginary in passing through equality, illustrating the fact that the imaginary roots occur in pairs. Further, he also presented rules to find a superior limit to the positive roots and to determine the approximate values of the *numerical roots*. The most fascinating result is his attempt to find a rule by which the number of roots can be determined by comparing the signs of certain fractions. Indeed, he was aware that his rule was not universal. This problem was handled and discussed by Campbell, Maclaurin, Euler and others. Subsequently, in 1865 Sylvester succeeded in proving the general result.

This article is in no way sufficient to gauge the strength and versatility of Newton as a mathematician. He had summed up an assessment of his work in the following words; *"I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the sea-shore and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me"*

Suggested Reading

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