Some Elementary Examples from Newton's *Principia*

H S Mani

This article is aimed at undergraduate students to give them a taste of the *Principia*, Newton's famous book. I have selected some examples and results in mechanics; most of them are a part of the undergraduate curriculum.

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1. Introduction

In the opinion of many scientists, Newton was not just one of the great scientists but the greatest of all times. His work on mechanics, using the three laws of motion which he enunciated covers a very wide canvas. It culminated with the discovery of the universal law of gravitation. All this was presented by Newton in his famous book *Philosophiae Naturalis Principia Mathematica* popularly known as the *Principia.* S Chandrashekar has been quoted in the cover of his book *Newton's Principia for the Common Reader* describing Newton as a scientist whose work is unsurpassed and unsurpassable.

Much of the mechanics that we learn and teach at the undergraduate level is based on the concepts developed by Newton, though the proofs we use are analytical, using both differential and integral calculus. Newton used many geometrical methods extensively to derive the results in spite of his having discovered calculus. Geometry, judiciously used with limiting procedures, was one principal strategy used by Newton in the *Principia.*

The *Principia* presents an enormous range of applications, based on the three laws of motion and the law of gravitation. Kepler's laws of planetary motion including the motion of comets, discussion of three body problem,

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lunar theory studying the variation of lunar motion, theory of tides, precession of equinoxes and several more phenomena are covered in the book. The proofs in the *Principia* are not easy to follow as most of the mathematical arguments are in prose. Fortunately for us S Chandrashekar's book mentioned earlier has removed this barrier. He mentions in the prologue "With the impediments of the language and of syntax thus eliminated, the physical insight and the mathematical craftsmanship that invariably illuminate Newton's proof come sharply into focus."

2. Examples

As mentioned before, the examples given below show the power of using Newton's laws together with the geometrical relations obeyed by trajectories. Newton was interested in finding the force law given the trajectory of the particle. In order to do this he obtained some general results, the area theorem being one of them.

2.1 Area Theorem

Consider a particle moving under the action of a force directed towards or away from a point referred to as the centre of force and denoted by S in *Figure* 1. The particle sweeps equal areas in equal times. The proof that we learn in books on mechanics goes as follows: Let $\vec{r}$ represent the position of a particle of mass m, acted upon by a central force $\vec{F} = k\vec{r}$ where k can be a function of the distance r. Newton's law gives

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} = k\vec{r}. \tag{1}$$

Newton was interested in finding the force law given the trajectory of the particle.

On taking the vector product of the above equation with $\vec{r}$ one gets

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = k\vec{r} \times \vec{r} = 0. \tag{2}$$

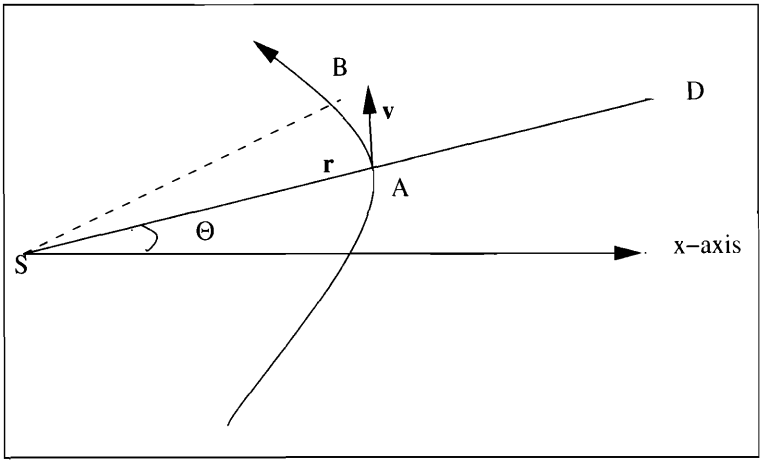


Figure 1. S is the centre of force, and v is the velocity of the particle at A.

This can be used to obtain

$$\frac{d(\vec{r} \times \frac{d\vec{r}}{dt})}{dt} = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0 \qquad (3)$$

implying

$$\vec{r} \times \vec{v} = \overrightarrow{constant} \qquad (4)$$

where $\vec{v} = \frac{d\vec{r}}{dt}$ is the velocity vector of the particle. Now let us use polar coordinates to describe the motion of the particle (see *Figure* 1). A particle from A reaches B, travelling along the trajectory AB, after a time interval δt. Thus $AB = v\delta t$ in the limit $\delta t \to 0$ (when $B \to A$).

Area of the triangle

$$\Delta ABS = SA \times AB \times \sin(\angle SAB) = rv\delta t \sin(\angle SAB).$$

One also has

$$|\vec{r} \times \vec{v}| = rv \sin(\angle BAD) = rv \sin(\angle SAB)$$

as $\angle BAD = 180^0 - \angle SAB$.

Thus the area swept by the particle in time Δt is equal to $|\vec{r} \times \vec{v}|\Delta t$ and is a constant.

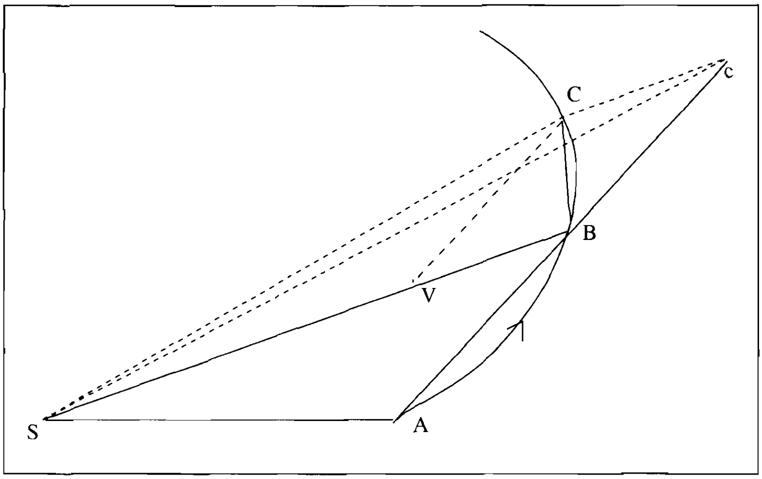


Figure 2. ABC is the trajectory of the particle, and S is the centre of force. AB = Bc. VBcC is a parallelogram.

Newton's Proof

Newton assumes that the particle is acted upon by impulses at equal intervals of time δt which keep it on the trajectory. In the absence of such an impulse the particle would move along a straight line path. This is shown in *Figure* 2.

The particle arrives at B from A after time δt. Had there been no impulse at B, it would have travelled to c with $\overrightarrow{Bc} = \vec{v}\delta t$, where $\vec{v}$ is its velocity at A. The impulse given along BS at B would push it to some point V The two displacements $\overrightarrow{BV}$ and $\overrightarrow{Bc}$ move it to C after a time $2\delta t$. (All the points S, A, B, V and c are in the same plane). Cc is parallel to BV and CV is parallel to Bc.

Now $\triangle SAB$ and $\triangle SBc$ have the same area since $AB = Bc = v\delta t$. $\triangle SCB$ and $\triangle ScB$ have the same area as the base SB is common and Cc is parallel to SB, implying that the altitudes of the two triangles are the same. Thus $\triangle SAB$ and $\triangle SCB$ have the same area proving the result. As a corollary, the area of the $\triangle SAB = v_A(\delta t)p_A$ and the area of the $\triangle SCB = v_B(\delta t)p_B$, where p_A, p_B are the lengths of the perpendiculars from S to AB and BC respectively. Further v_A, v_B are the velocities of the particle at A and B respectively The conservation of angular momentum

$$v_A p_A = v_B p_B$$

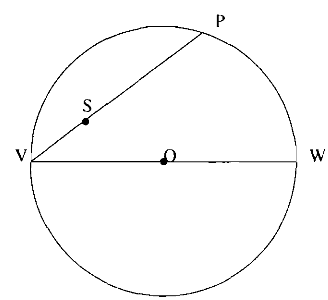
follows from the equality of the areas of the triangles $\triangle SAB$ and $\triangle SCB$.

Newton also observed the converse – if a particle travels in a planar trajectory and traverses equal areas in equal times the particle is acted upon by a central force, that is, the force is directed towards a fixed point (for an attractive force). This follows by noting the following: Given that the areas of the two triangles ΔSBC and ΔSAB are equal, the areas of the triangles ΔSBc and ΔSBC are equal as $AB = Bc$. This implies that the direction of the impulse is along cC which, being parallel to BS, leads to a central force directed towards S as $\delta t \to 0$.

2.2 *Circular Motion with Centre of Force Anywhere in the Circle*

Newton found the law of force for different orbits (circular, elliptical, parabolic, hyperbolic, spiral, etc.) for different possible choices of centres of force. For example if the orbit is elliptical and the centre of force is the centre of the ellipse (intersection of the major and minor axes of the ellipse), the force law is simple harmonic, i.e., force proportional to the distance from the centre. The most famous law, of course, is the inverse square law for elliptical orbits with the centre of force being one of the foci of the ellipse.

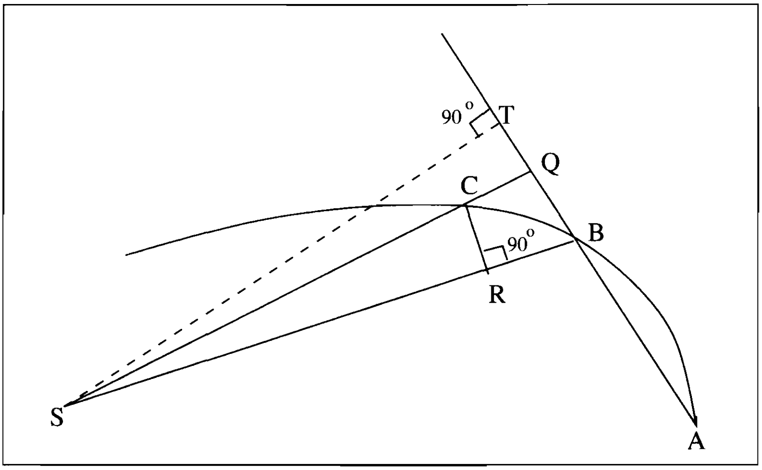
Figure 3. PS is extended to meet the circle at V. VOW is the diameter of the circle.



Another example is a particle moving in a circular orbit with the centre of force chosen as any point S inside the circle. This is shown in *Figure* 3. The result is

$$\textit{force} \propto \frac{1}{(SP)^2(PV)^3}, \tag{5}$$

Figure 4. ST is ⊥ to ABQ, CR is ⊥ to SB.



where V is the point on the circle when the line PS is extended (see *Figure* 3).

These results were obtained by the use of geometry and application of the laws of motion. The central result needed is the following:

Consider a particle moving along the orbit ABC under the action of a central force with centre at S as shown in *Figure* 4. In the notation of *Figure* 4, Newton showed that

$$C.F. \propto \frac{(QC)}{(SB)^2(CR)^2}, \tag{6}$$

where $C.F.$ stands for centripetal force.

Newton's arguments are reproduced below: ABQ would be the path of the particle if it moved as a free particle, in the absence of the central force. Let it arrive at Q from B after a time δt. The displacement $QC = \frac{1}{2}\times$(acceleration due to the central force) $\times(\delta t)^2$ where we have used the laws of motion for constant acceleration as the time interval is short. Thus

$$\overrightarrow{BC} = \overrightarrow{BQ} + \overrightarrow{QC} = \vec{v}\delta t + \frac{1}{2}\vec{a}(\delta t)^2 \tag{7}$$

where $\vec{v}$ is the initial velocity and $\vec{a}$ is the acceleration at B, which is proportional to the centripetal force.

Thus

$$C.F. \propto \frac{2QC}{(\delta t)^2}. \tag{8}$$

Further the area of the $\triangle SBC$ is equal to

$$\frac{1}{2}(SB)(CR) = K\delta t, \tag{9}$$

where K is a constant. Here the area theorem has been used which states that area swept by the particle is proportional to the time. Using (8) and (9),

$$C.F. \propto 2\frac{QC}{(\delta t)^2} = \frac{8(QC)K^2}{(SB)^2(CR)^2}. \tag{10}$$

This is the desired result (apart from $8K^2$ which can be absorbed in the constant of proportionality). The same can be put in a slightly different form: $\triangle STB$ (ST is $\perp$ to the line ABQ) and $\triangle CRB$ (CR is $\perp$ to SB) in *Figure* 4 are similar as $C \to B$ and $B \to A$. In this limit the line BT becomes tangent to the trajectory at B and $\angle SBT = \angle RBC$. Since the triangles $\triangle STB$ and $\triangle CRB$ are right-angled, they are similar and in this limit

$$\frac{ST}{SB} = \frac{CR}{CB}. \tag{11}$$

Substituting this in (6), we obtain the following general result

$$C.F. \propto \frac{QC}{(ST)^2(CB)^2}. \tag{12}$$

Using the geometrical properties of various curves very ingeniously, Newton obtained the relevant laws. This needed knowing theorems associated with conic sections, which were well established by then. Having established

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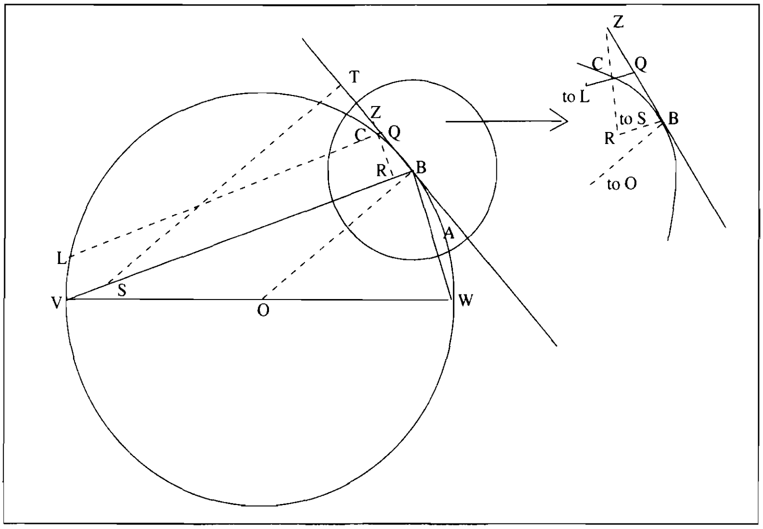

Figure 5. O is the centre of the circular trajectory. LQ is parallel to SB, ST is $\perp$ to BQZ; line CR is extended to meet BQ at Z.

a general formula we will apply it to the example of a particle moving in a circle with a centre of force anywhere inside the circle as in *Figure* 5.

As QC is parallel to RB, the triangles $\triangle ZRB$ and $\triangle ZCQ$ are similar. Further $\angle ZBO = 90^0$ and since $\triangle VBO$ is isosceles, $\angle RZB = \angle VBO = \angle BVO$. This means the triangles $\triangle VBW$ and $\triangle ZRB$ are similar. The similarity of these three triangles ($\triangle ZRB$, $\triangle ZCQ$ and $\triangle VBW$) implies

$$\frac{WV}{BV} = \frac{BZ}{RZ} = \frac{QZ}{ZC} = \frac{BZ - QZ}{RZ - ZC} = \frac{BQ}{RC}, \tag{13}$$

$$i.e., \; (RC)^2 = (BQ)^2 \frac{(BV)^2}{(WV)^2}. \tag{14}$$

The chords LC and BQ can be thought of as intersecting chords of a circle with the intersection point lying outside. (Here BQ is a tangent and is a special case of a chord in which the two intersecting points with the circumference coincide.) Using the following well known theorem in geometry

$$(BQ)^2 = (QL)(QC), \tag{15}$$

equation (14) becomes

$$(RC)^2 = (QL)(QC)\frac{(BV)^2}{(WV)^2}. \tag{16}$$

Using this in equation (10) the following result is obtained

$$C.F. \propto \frac{(WV)^2}{(SB)^2(BV)^3}. \tag{17}$$

As WV is the diameter of the circle, it is a constant and can be absorbed in the constant of proportionality. Note that if S lies on the circle, S and V coincide and the force law becomes

$$C.F. \propto (SB)^{-5} \tag{18}$$

2.3 *Revolving Orbits*

Let

$$r(\phi) = f(\phi) \tag{19}$$

describe the closed planar orbit, in terms of polar coordinates r, ϕ, of a body under the action of a central force. As a concrete example one could think of an elliptic orbit under the action of an inverse square law force with the centre of force at one of the foci. For such a case $r(\phi) = r(\phi + 2\pi)$. We will refer to this as the *fixed orbit.*

If the orbit is perturbed by external forces, in general the particle will not follow a closed path.

If the orbit is perturbed by external forces, in general the particle will not follow a closed path. One possible motion is a slow revolution of the major axis so that the particle describes an angle larger than 360^0 as it completes one revolution. The trajectory of the revolving orbit is given by

$$r'(\phi') = r(\phi), \tag{20}$$

where $\phi' = \alpha\phi$, α being a constant. Such an orbit will obey $r'(\phi' + 2\pi\alpha) = r(\phi + 2\pi) = r(\phi)$ and hence will

Newton developed the general theory for revolving orbits and worked out several special cases.

come back to the original position in terms of the fixed orbit at an angle different from 360^0.

For the fixed orbit, the area covered per unit time is given by

$$\frac{1}{2}r^2(\phi)\frac{d\phi}{dt} = h \quad (say), \tag{21}$$

where h is a constant. Then for the revolving orbit, the area covered per unit time is

$$\frac{1}{2}r'^2(\phi')\frac{d\phi'}{dt} = \alpha h \tag{22}$$

which is also constant and hence the force is central.

Newton developed the general theory for such orbits and worked out several special cases. The methods used are similar to the ones discussed in the earlier examples. He was particularly interested in the study of elliptical orbits which were nearly circular. He showed that if the force law is

$$r^{n-3} \tag{23}$$

then the angle described by the body when it moves from the farthest end of the major axis to the nearest end (central force is from one of the foci of the ellipse) is given by

$$\frac{180^0}{\sqrt{n}}. \tag{24}$$

$n = 1$ corresponds to the inverse square law for which the angle described is exactly 180^0. For a complete revolution the angle turned would be twice the value.

He generalised it to the case when the force law is

$$br^{m-3} + cr^{n-3} \tag{25}$$

and found the angle described to be

$$180^0 \left(\frac{b+c}{mb+nc} \right)^{\frac{1}{2}} \tag{26}$$

Newton applied this result to the slow motion of the apogee of the moon, which is 3.3^0 per complete revolution. Two possible central forces were considered to reproduce this result:

Case 1:

$$C.F. \propto \frac{1}{r^{2+\frac{4}{243}}} \simeq \frac{1}{r^{2.0165}}. \tag{27}$$

Case 2:

$$C.F. \propto \frac{1}{r^2} - .005595r. \tag{28}$$

Case 1: From equation (24), with $n = 0.9835$, the angle described for a complete revolution is

$$\frac{360}{\sqrt{0.9835}} = 363^0. \tag{29}$$

That is, it returns to the maximum distance from the centre of force after turning through an angle of 363^0.

Case 2: Using equation (26) with $b = 1, c = 0.005595$, the angle described for a complete revolution is

$$360\sqrt{\frac{1-0.005595}{1-4\times 0.005595}} = 363.1^0 \tag{30}$$

Newton considered the trajectory of the moon to be due to the gravitational forces of the earth and the sun. The earth–moon system moves in an elliptical (nearly circular) orbit around the sun. However the force exerted by the sun on the moon would be different from the one exerted on the earth and Newton conjectured that the second term in equation (30) arises from the correction needed to take this into account.

This formed a part of the analysis which culminated in the universal law of gravitation.

2.4 *N-body Problem*

This example is different from the previous ones, in the sense that the force law is assumed between N-bodies and an exact solution is obtained. This can be achieved by using the analysis learned in an undergraduate programme.

The force between any two mass points at $\vec{r_i}$ and $\vec{r_j}$ with masses M_i and M_j respectively $(i, j = 1, 2, \quad N)$ is assumed to be

$$-k^2 M_i M_j (\vec{r_i} - \vec{r_j}), \qquad i \neq j \tag{31}$$

where k is a constant.

Newton first obtained the solution for the case $N = 2$. Here we have

$$\frac{d^2\vec{r_1}}{dt^2} = -k^2 M_2 (\vec{r_1} - \vec{r_2}), \tag{32}$$

and

$$\frac{d^2\vec{r_2}}{dt^2} = -k^2 M_1 (\vec{r_2} - \vec{r_1}). \tag{33}$$

Choosing the frame in which the centre of mass, $\vec{G_2}$ is at rest and is its origin

$$\vec{G_2} = \frac{M_1\vec{r_1} + M_2\vec{r_2}}{M_1 + M_2} = 0. \tag{34}$$

Equations (31 and 32) become

$$\frac{d^2\vec{r_1}}{dt^2} = -k^2 (M_1 + M_2)\vec{r_1}, \tag{35}$$

$$\frac{d^2\vec{r_2}}{dt^2} = -k^2 (M_1 + M_2)\vec{r_2}. \tag{36}$$

The masses M_1, M_2 therefore execute simple harmonic motion with the same period about the centre of mass.

For $N = 3$, we have in the frame in which $\vec{G_2} = 0$,

$$\frac{d^2\vec{r_1}}{dt^2} = -k^2(M_1 + M_2)\vec{r_1} - k^2M_3(\vec{r_1} - \vec{r_3}) =$$

$$-k^2(M_1 + M_2 + M_3)\vec{r_1} + k^2M_3\vec{r_3}, \quad (37)$$

$$\frac{d^2\vec{r_2}}{dt^2} = -k^2(M_1 + M_2)\vec{r_2} - k^2M_3(\vec{r_2} - \vec{r_3}) =$$

$$-k^2(M_1 + M_2 + M_3)\vec{r_2} + k^2M_3\vec{r_3}. \quad (38)$$

Motion of M_3 is given by

$$\frac{d^2\vec{r_3}}{dt^2} = -k^2M_1(\vec{r_3} - \vec{r_1}) - k^2M_2(\vec{r_3} - \vec{r_2}) =$$

$$-k^2(M_1 + M_2)\vec{r_3}, \quad (39)$$

where (34) has been used. This can be rewritten as

$$\frac{d^2\vec{r_3}}{dt^2} = -k^2(M_1 + M_2 + M_3)\vec{r_3} + k^2M_3\vec{r_3}. \quad (40)$$

All the three equations can now be written in a symmetrical manner

$$\frac{d\vec{r_i}}{dt^2} = -k^2(M_1 + M_2 + M_3)\vec{r_i} + k^2M_3\vec{r_3} \qquad i = 1, 2, 3. \quad (41)$$

If we move to a frame which has an acceleration $k^2M_3\vec{r_3}$ with respect to the frame in which G_2 is at rest, we have (adding a pseudo force $-k^2M_3\vec{r_3}$)

$$\frac{d^2\vec{r_i}}{dt^2} = -k^2(M_1 + M_2 + M_3)\vec{r_i}. \quad (42)$$

This is the frame in which the centre of mass of the three particles $\vec{G_3}$ is at rest and is chosen as the origin, i.e.,

$$\vec{G_3} = \frac{M_1\vec{r_1} + M_2\vec{r_2} + M_3\vec{r_3}}{M_1 + M_2 + M_3}. \qquad (43)$$

Thus the three bodies execute simple harmonic motion about the centre of mass.

The method can be continued to obtain the solution for the general case.

3. Conclusion

Newton's *Principia* has an enormous variety of problems which are accessible to the undergraduate students. This is more so with the help of S Chandrasekhar's book *Newton's Principia for the Common Reader*.

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Suggested Reading

[1] S Chandrasekhar, *Newton's Principia for the Common Reader*, Clarendon Press, Oxford, 1995.

"Nature to [Newton] was an open book, whose letters he could read without effort."

– *Albert Einstein*
(quoted by G Simmons in "*Calculus Gems*" 1992)