Riemann and Theoretical Physics

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Joseph Samuel is a theoretical physicist and by natural inclination a classical mechanic. Over the years he has strayed into other fields like optics, general relativity and very recently DNA elasticity. A unifying theme in his work is differential geometry and topology in physics. He keeps moderately fit by raising and lowering indices and relaxes by playing semiclassical guitar.

Several mathematical disciplines such as differential geometry, topology and Riemann geometry on which Riemann left his mark have a major impact on physics today. This article describes some of these contributions of Riemann to physics.

Plane geometry has been studied since the ancient Greeks. The subject was consolidated in Euclid's *Elements*, which put down the main ideas in axiomatic form. Generations of mathematicians have marvelled at the architectural beauty of Euclidean geometry. A few years before. Riemann there began explorations of curved geometry. The subject of non-Euclidean geometry was arrived at independently by Gauss, Lobachevsky and Bolyai. This new geometry satisfied all of Euclid's axioms save one: the fifth postulate. One version of the fifth postulate states that given any figure there is a similar figure of any size. This postulate asserts that the Euclidean plane does not have an intrinsic scale. Scale independence is not true of curved geometries: the sphere has a radius. Non-Euclidean geometry was an important intellectual development: it established the logical independence of Euclid's fifth postulate and also introduced the idea of curved geometry. The sphere is a two dimensional surface of constant positive curvature and the saddle (or Lobachevsky space) is a two dimensional space of constant negative curvature. Both have an intrinsic scale related to the curvature.

Gauss studied curved two dimensional surfaces embedded in three dimensional flat space. He recognised the difference between properties intrinsic to the surface and extrinic ones, which depended on the embedding.

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Accordingly, Gauss introduced two notions of curvature, the mean curvature, which depended on the embedding space and what is now called "Gaussian curvature" The mean curvature measures how the surface varies in the embedding space and is also called the extrinsic curvature. For instance, the cylinder (the surface of a pipe) has extrinsic curvature, but it is not intrinsically curved (it can be cut and flattened on a piece of paper). He emphasised the intrinsic properties as worthy of study and proved that the "Gaussian curvature" is an intrinsic quantity: it can be determined by measurements made entirely within the surface. Gauss was so impressed by this theorem that he called it the *Theorema Egregium.* This latin phrase can be loosely and colloquially translated into modern English as "Heap Big Theorem"

Gauss emphasized that the intrinsic geometry of a surface was worthy of study.

Riemann's approach to curvature was far more general than Gauss's. In a small region of space (small compared to the scale of the curvature) one can adopt a local notion of parallelism: a rule for comparing vectors at different points. It turns out that on a curved space this local notion does not integrate to produce a global notion of parallelism. This is best illustrated by using the example of the surface of the Earth. In everyday life, we do not bother about the curvature of the Earth (since the Earth is much bigger than our cities) and some planned cities (Salt Lake City, Utah, USA is a good example) have a Cartesian grid of streets laid out straight like a chess board. We take pains to ensure that the Main roads are directed "parallel" to each other. Like wise, the cross roads are directed "parallel" to each other and meet the Main roads at right angles. If you start on a Main road and take four left turns of ninety degrees (walking say, five blocks between turns) you will be back on a Main road. However, if the city grows so that it covers a good fraction of the Earth's surface (say 1/8), this parallelism is no longer possible. (Try sticking small bits of graph paper on an octant of

Riemann's concept of curvature was more general and therefore to relevant to Einstein's relativity.

a sphere maintaining parallelism!) You will find that if you start on a Main road and take four left turns (each after traversing a quarter of the Earth's circumference), you end up on a cross road. The grid has rotated by a right angle! Riemannian curvature is defined using the non-integrability of parallelism. Riemann's definition of curvature has considerable advantage over Gauss's: it is manifestly intrinsic. The *Theorema Egregium* is no longer a theorem but an obvious fact. Riemann's ideas generalise easily to any number of dimensions. They also work for metrics of Lorentzian signature. This turned out to be important for General Relativity. It was already clear from special relativity that space-time in the absence of gravity (Minkowski space-time) had a geometry of Lorentzian signature. This follows from the fact that in special relativity it is the interval $x^2+y^2+z^2-c^2t^2$ which is of primary interest, as opposed to the (squared) length $x^2 + y^2$ of Euclidean geometry. What Einstein needed to describe the gravitational field was curvature in four dimensional space-time, *i.e* space-time with a four dimensional Lorentzian metric. Such abstraction was not easy to achieve based on Gauss' line of attack using embedding in Euclidean space.

Einstein's General Theory of Relativity went on to acquire a life of its own, predicting many physical effects that could be measured in the Solar System, revolutionising Cosmology and Astrophysics. The theory also predicts Black Holes, which continue to stretch our imagination and defy our understanding. Many ideas which now form the General Theory of Relativity were anticipated by Riemann. He realised that space may be curved and that this is a question which has to be settled by experiment, not philosophical speculation. He even conceived of the possibility that space may be discrete, an idea which is only now coming of age in quantum gravity; it is believed by some that space-time is discrete at the Planck scale of 10^{-33} cm. Evidently, Riemann the mathematician was well plugged in to the physical world. He suggested (seven years before Maxwell's equations were written) that electromagnetic interactions propagate at the speed of light! For his habilitation lecture (a requirement by the university for an aspiring teacher) Riemann had to give three possible topics. Two of these were on electricity! But it was the third on the foundations of geometry that Gauss picked. This was the famous lecture in which Riemann's ideas on differential geometry like n-dimensional manifolds were put forward. It is fair to say that apart from Gauss (who had the habit of publishing far less than he knew) no one in the audience appreciated the full depth of the ideas proposed there.

Riemann's name appears so often in physics texts that we sometimes forget it is a proper noun!

Another subject to which Riemann contributed is complex analysis. Complex analytic techniques are now widely used in physics and engineering. Two dimensional problems in potential theory, elasticity and fluid dynamics are routinely addressed using complex analytic techniques like conformal mappings. It is interesting that Riemann first encountered conformal mappings when studying a problem related to the heat equation! The Riemann zeta function appears in physics texts dealing with the statistical distribution of Fermi and Bose particles. It is also used in renormalisation in quantum field theory to make sense of divergent infinite sums. One uses analytic continuation to give a meaning to the sum by evaluating the zeta function on a different Riemann sheet.

Riemann's approach to complex analysis was geometric and intuitive. Many of his contemporaries viewed complex analysis from an algebraic point of view. Riemann's geometric point of view resulted in advances in differential geometry and topology. Topological ideas are extremely intuitive and at the same time very hard to formalise. Leibniz (1646-1716) was one of the first mathematicians to study "*Analysis Situs*" as topology used to be called, but he was unable to interest physicists of his time (like Christiaan Huygens) in the subject. Physicists were more interested in quantification of physical ideas and the time was not yet ripe of the qualitative reasoning that marks topology. Much later Euler (see Crossing Bridges, in Suggested Reading below) played with topological ideas in his famous solution of the problem of the Bridges of Königsberg.

When Riemann studied complex analysis, his unique approach led him to the idea of a Riemann Surface. This led to considerable advances in topology. He was followed by Betti and then by Poincare, who laid the foundations of topology in its modern form. The twentieth century has seen a rapid development of the subject, culminating recently (2006) in the solution of the Poincaré conjecture by Grigori Perelman.

Physics has gained from these developments in mathematics. Topology is used to classify defects in liquid crystals and to describe vortices in superconductors and magnetic monopoles. Differential geometry is used in the study of gauge theories and gravitation. We model space-time as a real differentiable manifold. Complex manifolds (Calabi-Yau spaces) are used by String theorists. Theoretical physicists these days glibly speak of ten dimensional spaces. It is important to remember that in Riemann's day, higher dimensions were viewed with suspicion even by mathemticians. The idea of space and time forming a four dimensional continuum was not then in existence. Riemann's approach was marked by generality and abstraction and paved the way for the major intellectual developments of theoretical physics in our time.

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Suggested Reading

[1] M Monastyrsky Riemann, *Topology and Physics*, Birkhauser, Boston, 1987.

[2] Joseph Samuel, Crossing Bridges, *Resonance*, Vol.4, p.23, 1999.

[3] S Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, 1972.

[4] James Hartle, *Gravity: An Introduction to Einstein's General Theory of Relativity*, Pearson Education, New Delhi, 2002.

[5] C Nash and S Sen, *Topology and Geometry for Physicists*, Academic Press, 1983.

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