$s > 1$. (See Appendices for mathematical details). Riemann extended this function by analytic continuation to the entire complex plane as a meromorphic function with only a simple pole at $s = 1$ with residue 1. The enormous interest in this series was due to a result published by Riemann in 1859, where he obtained the formula for the number of primes up to a given number. Here he showed that this depends on the zeros of the Zeta function.

The Riemann hypothesis can be stated thus: "The nontrivial zeros of $\zeta(s)$ have real part equal to 1/2."

Riemann considered that function of the complex variable t defined by

$$\xi(t) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

with $s = \frac{1}{2} + it$ and showed that $\xi(t)$ is an even entire function of t, whose zeros have imaginary parts that lie between $-i/2$ and $i/2$, that is $0 < \Re(s) < 1$.

He also made the observation that in the range between 0 and T, the function $\xi(t)$ has about $T/2\pi \log(T/2\pi) - T/2\pi$ zeros. Riemann made the famous remark (in German) "Indeed, one finds between these limits about that many real zeros and it is likely that all zeros are real"

The function $\zeta(s)$ has zeros at the negative even integers $-2, -4, -6, \ldots$ and these are called the trivial zeros. The other zeros are the complex numbers $\frac{1}{2} + i\alpha$, where α is a zero of $\xi(t)$. In terms of the function $\zeta(s)$, the Riemann hypothesis can be stated thus:

"The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$."

"In the opinion of many mathematicians, the Riemann hypothesis and its extension to general class of L functions, is probably the most important open problem in pure mathematics today."

3. Prime Numbers and the Zeta Function

The connection between prime numbers and the Zeta function appeared for the first time in the work of Leonhard Euler in 1748. This was done with the famous Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

valid for $\Re(s) > 1$, where the product is taken over all primes p.

The problem of the distribution of prime numbers was initially investigated by Gauss at the end of the eighteenth century. In a letter to the astronomer Hecke in 1849, he stated that he had found in his early years that the number $\pi(x)$ of primes up to x is well approximated by the function

$$Li(x) = \int_0^x \frac{dt}{\log t}.$$

It was the Russian mathematician Tchebychev, who in 1848, presented his results to the Academy of St. Petersburg on the distribution of prime numbers. He proved that the integral formula

$$\zeta(s) - 1 - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) e^{-x} x^{s-1} dx$$

led to the result that $(s - 1)\ \zeta(s)$ has limit 1, as well as finite derivatives of all orders as s tends to 1 from the right. From this he deduced an asymptotic formula for $\pi(x)$ by means of a finite sum $\sum_k \frac{a_k x}{(\log x)^N}$ up to an order $O\left(\frac{x}{(\log x)^N}\right)$, where $a_k = (k-1)!$, $k = 1, 2, \quad , N - 1$. This is precisely the asymptotic expansion of the function $Li(x)$, which Gauss had suggested.

What Riemann did was to use the integral formula valid for $\Re(s) > 1$, namely,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{e^{-x}}{1 - e^{-x}} \right) x^{s-1} dx$$

and then deform the contour of integration in the complex plane so as to obtain a representation valid for all s. Riemann then wrote the logarithm of the Euler product for $\Re(s) > 1$, namely

$$\frac{1}{s}\log\zeta(s) = \int_1^\infty \Pi(x)x^{s-1}dx,$$

where $\Pi(x) = \pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi\ (\sqrt[3]{x}) + .$ By Fourier inversion, he was able to express $\Pi(x)$ as a complex integral and compute it using the calculus of residues. The residues occur at the singularities of $\log\zeta(s)$ at $s = 1$ and at the zeros of $\zeta(s)$. Finally an inversion formula expressing $\pi(x)$ in terms of $\Pi(x)$ gives Riemann's formula.

4. Recent Work

Though no proof of the Riemann hypothesis has yet been given, there is strong numerical evidence that it holds. Riemann himself had computed the first few nontrivial zeros of the Zeta function. Thanks to modern computational expertise, it has been shown that the first 1.5 billion zeros of $\zeta(s)$ arranged by increasing positive imaginary part are simple and satisfy Riemann hypothesis. Further, it is known that more than forty per cent of nontrivial zeros of $\zeta(s)$ are simple and satisfy Riemann hypothesis.

The Riemann hypothesis has been a central problem of pure mathematics, not only because of its importance with regard to the distribution of prime numbers, but also because it is the prototype of a general class of functions, called L- functions. They are Dirichlet series with a suitable Euler product and are expected to satisfy a Riemann hypothesis.

The Riemann hypothesis for $\zeta(s)$ does not seem to be any easier than that for the Dirichlet L-functions. This has lead to the view that its solution may require the use of much more sophisticated techniques and entirely new ideas.

Appendix 1. Analytic Continuation of $\zeta(s)$

The Zeta function can be written as $\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^s}$, where $s = \sigma + it$.

Theorem 1. The series for $\zeta(s)$ converges absolutely for $\sigma > 1$. The convergence is uniform in the half plane $\sigma > 1 + \delta$, $\delta > 0$, so $\zeta(s)$ is an analytic function of s in the half plane $\sigma > 1$.

Proof:

$$\sum_{n=0}^{\infty} |(n+1)^{-s}| = \sum_{n=0}^{\infty} (n+1)^{-\sigma} \leq \sum_{n=0}^{\infty} (n+1)^{-(1+\delta)}$$

Theorem 2. The Zeta function can be represented for $\sigma > 1$ by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\frac{e^{-x}}{1 - e^{-x}} \right) x^{s-1} dx$$

Proof: Suppose s is real. Then the Gamma function is given by

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx.$$

Let $x = (n+1)y$, then

$$\Gamma(s) = (n+1)^s \int_0^{\infty} e^{-(n+1)y} y^{s-1} dy$$

$$\Rightarrow \quad (n+1)^{-s} \Gamma(s) = \int_0^{\infty} e^{-ny} e^{-y} y^{s-1} dy.$$

Summing over all n,

$$\zeta(s)\Gamma(s) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-ny} e^{-y} y^{s-1} dy,$$

the series on the right hand side being convergent if $s > 1$. In order to interchange the sum and the integral, one could regard the integral as a Lebesgue integral and use Levi's convergence theorem to get

$$\zeta(s)\Gamma(s) = \int_0^\infty \sum_{n=0}^\infty e^{-ny} e^{-y} y^{s-1} dy.$$

If $y > 0$, then $0 < e^{-y} < 1$ and hence

$$\sum_{n=0}^\infty e^{-ny} = \frac{1}{1-e^{-y}}.$$

This gives

$$\zeta(s)\Gamma(s) = \int_0^\infty \left(\frac{e^{-y}}{1-e^{-y}}\right) y^{s-1} dy$$

almost everywhere on $[0,\infty)$. This holds, in fact, everywhere except at 0, so for all $s > 1$,

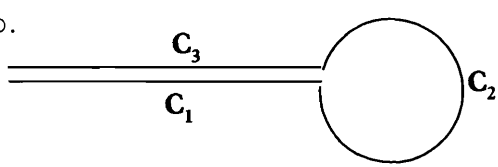
$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{e^{-y}}{1-e^{-y}}\right) y^{s-1} dy.$$

Moving over to the complex plane, the integral converges uniformly in every strip $1+\delta \leq \Re(s) \leq c$, where $\delta > 0$ and therefore represents an analytic function in every such strip, hence also in the half plane $\Re(s) > 1$. By analytic continuation, it holds for all s, with $\Re(s) > 1$.

Appendix 2. A Contour Integral Representation of the Zeta Function

Consider a loop C consisting of three parts C_1, C_2, C_3. C_2 is a positively oriented circle of radius $c < 2\pi$ about the origin and C_1, C_3 are the lower and upper edges of a cut in the z-plane along the negative real axis

On C_1, $z = re^{-i\pi}$ and on $C_3, z = re^{i\pi}$, where r varies from c to ∞.



Theorem 3. The function defined by the contour integral

$$I(s) = \frac{1}{2i\pi}\int_C \frac{z^{s-1}e^z}{1-e^z}dz,$$

is an entire function of s. Also $\zeta(s) = \Gamma(1-s)I(s)$, if $\Re(s) > 1$.

Proof:

$$2\pi i I(s) = \int_{C_1} + \int_{C_2} + \int_{C_3} z^{s-1}g(z)dz,$$

where $g(z) = \frac{e^z}{1-e^z}$.

On $C_1, C_3, g(z) = g(-r)$ and on $C_2, z = ce^{i\theta}$

$$\begin{aligned} &2\pi i I(s) \\ = &\int_\infty^c r^{s-1}e^{-is\pi}g(-r)dr + i\int_{-\pi}^{\pi} c^{s-1}e^{(s-1)i\theta}ce^{i\theta}\ g(ce^{i\theta})d\theta \\ &+ \int_c^\infty r^{s-1}e^{is\pi}g(-r)dr \\ = &\ 2i\sin(\pi s)\int_c^\infty r^{s-1}g(-r)dr + ic^s\int_{-\pi}^{\pi} e^{is\theta}g(ce^{i\theta})d\theta \\ \Rightarrow &\ \pi I(s) = \sin(\pi s)I_1(s,c) + I_2(s,c). \end{aligned}$$

Letting $c \to 0$, we have

$$\lim_{c\to 0} I_1(s,c) = \int_0^\infty \frac{r^{s-1}e^{-r}}{1-e^{-r}}dr = \Gamma(s)\zeta(s),$$

if $\sigma > 1$ and $\lim_{c\to 0} I_2(s,c) = 0$. $g(z)$ is analytic in $|z| < 2\pi$, except for a first order pole at $z = 0$. $zg(z)$ is analytic everywhere inside $|z| < 2\pi$ and hence $|g(z)| < A|z|$ for $|z| = c < 2\pi$, where A is a constant.

$$|I_2(s,c)| \le \frac{c^\sigma}{2}\int_{-\pi}^{\pi} e^{-t\theta}\frac{A}{c}d\theta \le Ae^{|t|\pi}c^{\sigma-1}$$

If $\sigma > 1$ and $c \to 0$, we have $I_2(s,c) \to 0$. Hence $\pi I(s) = \sin(\pi s)\Gamma(s)\zeta(s)$. Since $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we have

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi}\int_C \frac{z^{s-1}e^z}{1-e^z}dz.$$

Appendix 3. The pole at $s = 1$ of $\zeta(s)$

Theorem 4. The function $\zeta(s)$ is analytic for all s, except for a simple pole at $s = 1$ with residue 1.

Proof: Since $I(s)$ is entire, the only possible singularities of $\zeta(s)$ are the poles of the Gamma function $\Gamma(1-s)$, namely $s = 1, 2, 3, ..$ But $\zeta(s)$ is analytic for $s = 2, 3,$ So $s = 1$ is the only possible pole of $\zeta(s)$.

If s is an integer, the integrand in the contour integral of $I(s)$ takes the same value on C_1 as on C_3 and hence the integrals on C_1 and C_3 cancel.

$$\begin{aligned} I(s=1) &= \frac{1}{2i\pi}\int_{C_2} \frac{e^z}{1-e^z}dz \\ &= \operatorname*{Residue}_{z=0} \frac{e^z}{1-e^z} = \lim_{z\to 0}\frac{ze^z}{1-e^z} = -1 \end{aligned}$$

To find the residue of $\zeta(s)$ at $s = 1$,

$$\begin{aligned} \lim_{s\to 1}(s-1)\zeta(s) &= -\lim_{s\to 1}(1-s)\Gamma(1-s)I(s) \\ &= -I(s=1)\lim_{s\to 1}\Gamma(2-s) = \Gamma(1) = 1 \end{aligned}$$

$\zeta(s)$ has a simple pole at $s = 1$ with residue 1.

Appendix 4. Trivial zeros of $\zeta(s)$

For all s, $\sigma > 1$,

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{s\pi}{2}\right)\zeta(s)$$

Taking $s = 2n + 1$, $n = 1, 2, 3, .,$ the factor $\cos\left(\frac{s\pi}{2}\right)$ vanishes and we get the trivial zeros of $\zeta(s)$.

$$\zeta(-2n) = 0, \quad n = 1, 2, 3\,.$$

Also $\zeta(s)$ satisfies the functional equation

$$(\pi)^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = (\pi)^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

Suggested Reading

[1] E Bombieri, ***The Riemann Hypothesis, The Millennium Prize Problems***, eds. J Carlson, A Jaffe and A Wiles, Clay Mathematics Institute and American Mathematical Society, 2006.

[2] T M Apostol, ***Introduction to Analytic Number Theory***, Springer, 1989.

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