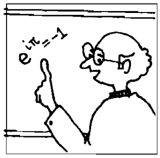

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Archimedes : Bathtub Academic *par excellence*

Stability and control of floating bodies is a major aspect of navigation from ancient to modern times. It is indeed remarkable that Archimedean concepts continue to guide watercraft designers even after 2200 years! Extending Euclidean concept to solids, Archimedes solved the problem of a paraboloid floating stably but with its base inclined to the water surface. Here, we revisit Archimedes and present stability maps (A-maps) for paraboloid, solid cone and a prism. For a specific combination of geometry and specific gravity, these A-maps predict whether a solid floats vertically or in a tilted fashion; or, whether it is partially submerged.

1. Introduction

Have you ever wondered how pans and pails sink in water as they get filled up? Or, how engineers design boats and ships to stay afloat in the sea? Have you ever noticed how toys and sundry float in a baby bathtub? Would you be able to name the Sicilian genius who transformed his bathtub to one of the most advanced laboratories and whose ideas have survived the test of time like his contemporary Euclid? You would most certainly be able to name the scientist-philosopher who discovered buoyancy

Keywords
Hydrostatics, stability of floating bodies, metacentre, centre of buoyancy, paraboloid, Archimedean map (A-map), moment of inertia.

screaming *Eureka*! Well, the answer is obviously Archimedes. Schooled in ancient Alexandria, Euclid and Archimedes continue to inspire generations of students and teachers of mathematics, physics and engineering.

Archimedes literally immersed himself in a bathtub of ideas and inventions. Hydrostatics and hydrodynamics command a long and respectable history dating back to Archimedes and beyond. Navigation and commerce along rivers and canals and across oceans in the early history of human civilization created huge opportunities for a large number of merchants, poets, pirates, artisans, skilled workers and scientist-philosophers like Archimedes.

Archimedes literally immersed himself in a bathtub of ideas and inventions. Hydrostatics and hydrodynamics command a long and respectable history dating back to Archimedes and beyond.

It is quite interesting to note that analytical excellence demonstrated by Archimedes and Euclid was virtually non-existent before their time in the recorded history of science. It is even more surprising to add that it took another 1900 years for the world to witness the rebirth of mathematical analysis of natural phenomena through calculus invented by Leibniz and Newton.

Stability of different shapes of floating bodies in water is a fundamental requirement in the design of boats and ships. Consequently, the geometrical properties and the specific gravity of the floating bodies determine the stable configurations. In the case of symmetrical shapes, a stable vertical configuration is not guaranteed in all cases. Archimedes addressed this problem for the specific shape of a right paraboloid floating with its base *above* the water surface. It is quite remarkable that Archimedes derived a mathematical criterion connecting specific gravity (s), the tilt angle and the geometrical parameter $\tan\phi$, where ϕ is the base angle at the rim of the paraboloid (shown in *Figure* 3, p.83). It is inconceivable to reconstruct his proof today without using modern calculus which was invented 1900 years after Archimedes' death.

Stability of different shapes of floating bodies in water is a fundamental requirement in the design of boats and ships. Consequently, the geometrical properties and the specific gravity of the floating bodies determine the stable configurations.

There is a popular website on Archimedes actively maintained by Chris Rorres www.math.nyu.edu/~crorres/Archimedes/Floating/floating.html

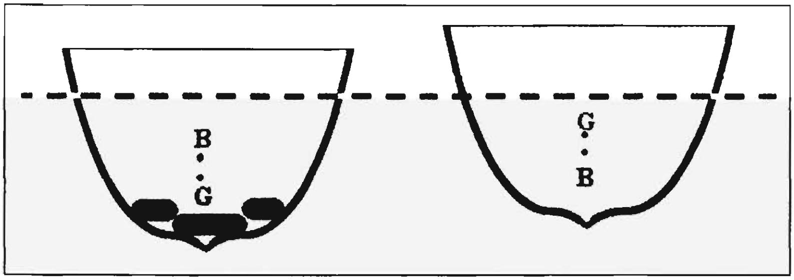


Figure 1. Stable configurations.

This website includes animations and illustrations on this ancient problem of stability of floating bodies which continues to challenge teachers and students of hydrostatics and naval engineering. It is unfortunate that most modern texts neglect advanced topics in hydrostatics owing to advanced mathematical skills required pertaining to solid geometry and calculus.

2. Stability of Floating Bodies

Archimedes formulated the laws of floatation. The buoyant upthrust balances the weight of the floating body. Archimedes showed that the buoyant force equals the weight of the displaced liquid. The buoyant force resultant acts at a point called the centre of buoyancy B. This is similar to the concept of centre of gravity G. The floating body is in equilibrium when the line joining GB is normal to the water surface. This is a necessary condition for stability but insufficient to assess stability of a floating body. It is important to note that G need not necessarily be below B.

Stability denotes the ability of a body to retain its position and orientation in space. Thus when the body is disturbed *slightly*, it returns to the original configuration. The amount of disturbance is assumed to be small while examining stability of a body. Assessing stability of a floating body requires additional concepts (see [1]).

Stability denotes the ability of a body to retain its position and orientation in space. Thus when the body is disturbed *slightly*, it returns to the original configuration.

Consider a cylinder of specific gravity s, height H and radius R floating in a liquid. The immersion is sH. When the cylinder is depressed slightly it bounces up for small values of H. Similarly when it is tilted slightly from its vertical configuration, it returns

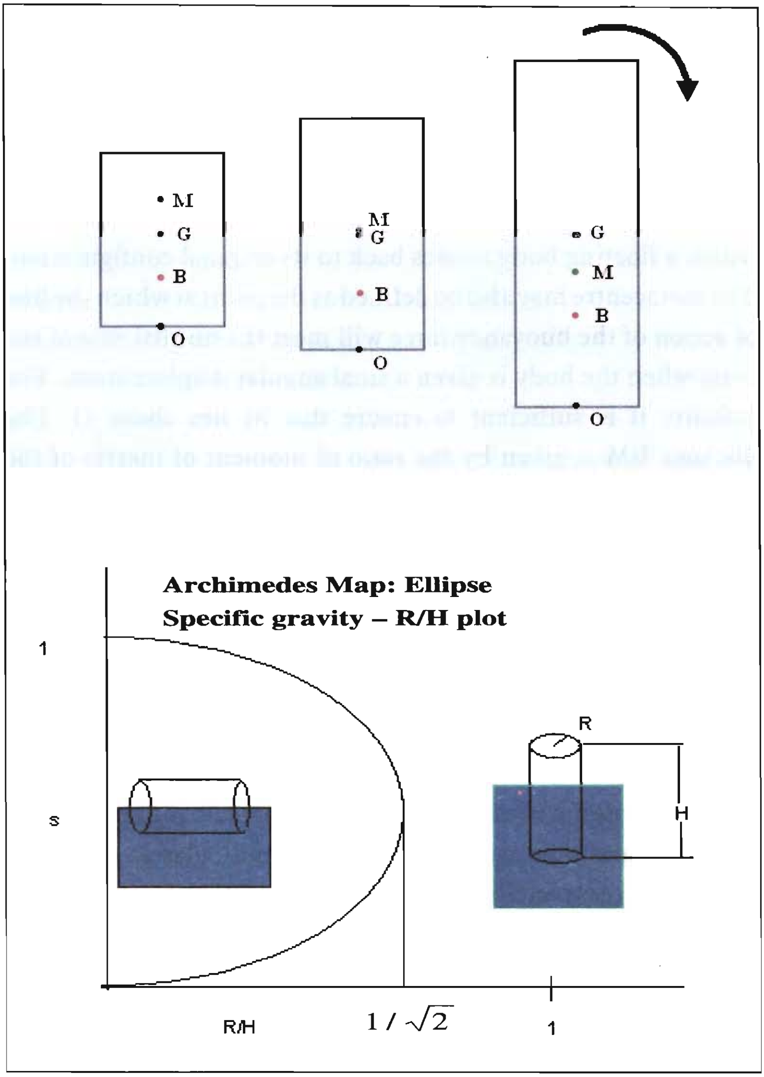


Figure 2. Stability, metacentre and tumbling.

to vertical. This situation is not true for a taller cylinder. There is a critical height H_c, for each set of values of s and radius R, when the cylinder tumbles to a horizontal configuration. This is illustrated in *Figure* 2 which also includes the (s – R/H) plots showing the critical value of R/H for a given s below which the cylinder tumbles to a horizontal configuration. This gives the stable region in the s versus R/H ($=\alpha$) plane. It is convenient to refer to $s - \alpha$ plots as *Archimedean maps* (A-maps) in the sequel while discussing configurational transitions of floating objects.

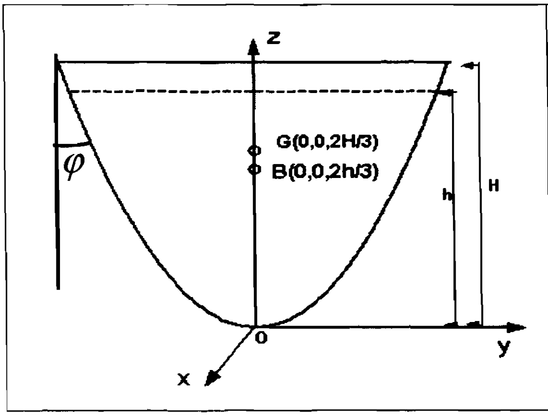
The stability analysis provides a means of classifying floating objects into plates and pencils. Plates float with their axis perpendicular to the water surface whereas pencils tumble to a horizontal configuration like a wooden log.

The unstable region when the cylinder tumbles to a horizontal configuration is a semi-ellipse of major axis $1/\sqrt{2}$ and minor axis 1/2. A similar situation develops for a floating paraboloid considered by Archimedes.

The stability analysis is based on the concept of *metacentre* about which a floating body rotates back to its original configuration. The metacentre may also be defined as the point at which the line of action of the buoyancy force will meet the normal axis of the body when the body is given a smal angular displacement. For stability it is sufficient to ensure that M lies above G. The distance BM is given by the ratio of moment of inertia of the cross section at the water surface (I) to the immersed volume of the floating body (sV). The stability condition stipulates $H^2/R^2 < 1/[2s(1-s)]$. It is interesting to note that H/R becomes infinite for $s = 0$ or 1. The first case of $s = 0$ implies extremely high density liquid approaching a solid. In this case a rigid cylinder can remain stable (neglecting elastic effects!). In this special case $H/R \sim 1/(2s)^{1/2}$. The latter case of $s = 1$ implies a neutrally buoyant cylinder which can assume an arbitrary orientation. This analysis provides a means of classifying floating objects into plates and pencils. Plates float with their axis perpendicular to the water surface whereas pencils tumble to a horizontal configuration like a wooden log.

Conventional teaching of hydrostatic stability is usually limited to vertical configuration about planes of symmetry though Archimedes developed the famous criterion for *asymmetric* configuration over 2200 years ago!

In this article, we revisit the Archimedes' criterion for a right paraboloid and develop Archimedean criteria for two other shapes namely, a cone and a parabolic prism. The latter shape models the case of floating vessels. Archimedean criteria for these shapes are illustrated through stability maps. In these maps the region of stable vertical equilibrium is demarcated from the region of stable, tilted equilibrium configurations. The concept of *metacentre* is explained for vertical stability. The tilted



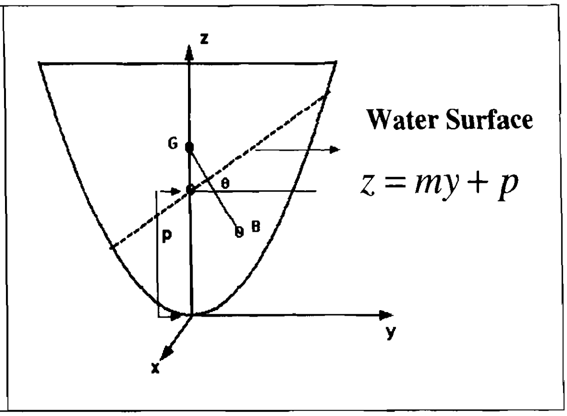


Figure 3 (left). Schematic of Configuration A.
Figure 4 (right). Schematic of Configuration B.

configuration is a stable configuration provided that the centre of gravity lies vertically above the centre of buoyancy. When the centre of gravity G lies vertically below the centre of buoyancy B, the floating body is unconditionally stable. This is normally the case for cargo ships and primitive watercraft such as dugouts and catamarans. Stability becomes a serious issue when the centre of gravity lies above the centre of buoyancy. This issue forms the theme of this article.

3. Archimedean Criterion

Archimedes considered a right paraboloid. A right paraboloid is a solid of revolution obtained by rotating a parabola about its axis. Paraboloidal shapes are used widely for antennae and solar collectors. (Archimedes however considered a solid paraboloid of height H and base radius R which is assumed to be lighter than water.) Three distinct configurations are possible when this paraboloid is floating in water depending on its specific gravity (s) and the aspect ratio of the body $\alpha = R/H$:

Vertical Configuration **A**: The base remains horizontal and stays above water (*Figure* 3).

Tilted Configuration **B**: The base is inclined with an angle of tilt θ to the water surface but stays above water as shown in *Figure* 4.

Submerged Configuration **C**: The base is partially submerged.

4. Vertical Configuration Analysis

The equation of the right paraboloid is

$$z = k\,(x^2 + y^2), \tag{1}$$

where x, y and z are the Cartesian coordinates with origin at the vertex of the paraboloid and z–axis is taken along the axis of the body as shown in *Figure* 3. The constant k is given by $k = H/R^2$ and it can be seen that $\tan\varphi = 2kR$. The water surface is parallel to the base of the paraboloid.

The volume of the paraboloid V is given by $V = \pi R^2 H/2$. The immersed volume $V_{IM} = \pi\, r^2 h/2 = sV$ implying $s^{1/2} = h/H$. The coordinates of the centre of gravity G are $(0, 0, 2H/3)$. The centre of buoyancy B is the centroid of the immersed volume with coordinates $(0, 0, 2Hs^{1/2}/3)$. The point G is above B on the same vertical axis which also represents the axis of the paraboloid. For this configuration we can locate the metacentre M by using the relation $\mathrm{BM} = I/V_{IM}$, as previously explained for the case of a floating cylinder. As in the case of the cylinder, the stable configuration of a shallow paraboloid is with its axis perpendicular to the water surface.

5. Tilted Configuration Analysis

When the paraboloid floats with its base completely above the water level in an inclined position, the volume of the water displaced remains same as that of the vertical configuration V_{IM}. Let us assume that the base is inclined to the water surface with an angle of tilt θ Without loss in generality we can fix the Cartesian coordinates to the vertex and the axis of the paraboloid as shown in *Figure* 4 and the water surface can be given by the inclined plane $z = k(\overline{m}y + \overline{p})$. Analytical expressions for the immersed volume V_{IM} and the buoyancy centre B can be obtained as

$$V_{IM} = \iiint_{V_{IM}} dV = \frac{\pi}{2} k \left(\overline{p} + \frac{\overline{m^2}}{4} \right)^2 ;$$

$$B_x = 0;\; B_y = \frac{1}{V_{IM}} \iiint_{V_{IM}} y\,dV = \bar{m}/2;$$
$$B_z = \frac{1}{V_{IM}} \iiint_{V_{IM}} z\,dV = \frac{k}{12}(5\bar{m}^2 + 8\bar{p}). \qquad (2)$$

For a given slope of the inclined water surface $m = \bar{m}k$, $\bar{p}$ can be obtained, using the fact that the volume of water displaced is the same as that of the vertical configuration, to be

$$\bar{p}k = r^2 - m^2/4 \cdot \qquad (3)$$

Archimedes showed that the angle of tilt is controlled by the specific gravity and geometry based on the fact that the line BG joining the points G and B is *perpendicular* to the water surface:

$$\frac{(B_z - \frac{2}{3}H)}{\frac{\bar{m}}{2}} = -\frac{1}{k\bar{m}} \qquad (4)$$

Using equations (2), (3) and (4), the equation relating the specific gravity s, aspect ratio α and the slope of the water surface with respect to the axis of the tilted paraboloid '$m = \tan\theta$' is derived as:

$$\frac{8}{3\alpha^2}(1 - s^{1/4})(1 + s^{1/4}) = 2 + m^2 \qquad (5)$$

This complex relationship is shown in three-dimensional space (s, ϕ, θ) by Rorres for a range of values $0 \le s \le 1$, ϕ : 0 to 90°. The limiting case of infinitely long paraboloid (i.e $\phi = 90°$) leads to two extreme values of specific gravity, namely, $s = 1$ and $s = (1/5)^4 = 1/625$. This is better explained using a modified A-map, ($s^{1/4} - \alpha$) plot, shown in *Figure* 5, demarcating the transitions from configuration A to B shown by the solid line, and from B to C shown by the dotted line. For better graphical elegance the fourth root of s forms the ordinate in this A-map. The A–B

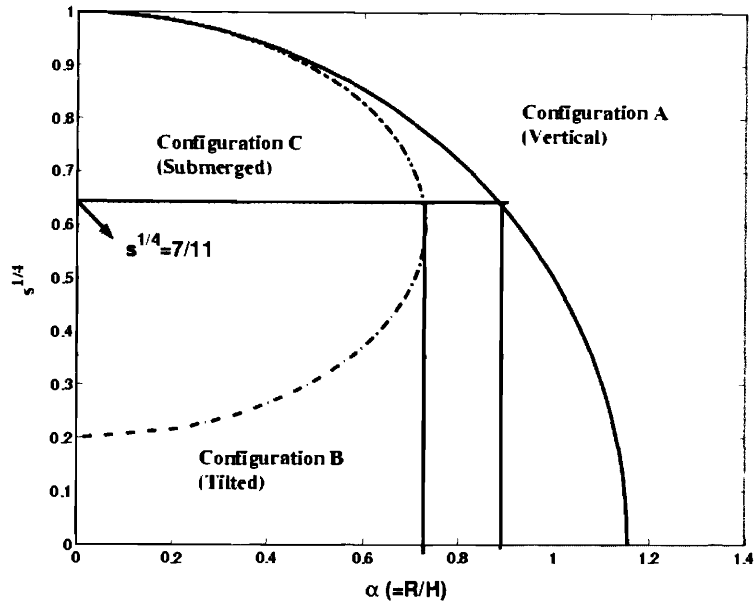


Figure 5. A-Map of a paraboloid.

transition is given by the equation obtained by putting $m = 0$ in equation (5):

$$s^{1/4} = \left(1 - \frac{3}{4}\alpha^2\right)^{1/2} \tag{6}$$

The B–C transition occurs when the water surface cuts the rim of the paraboloid. This constrains the parameters $\bar{m}$ and $\bar{p}$ to satisfying a fixed immersed volume V_{IM} yielding:

$$\frac{3}{2}\alpha^2 = (1 - s^{1/4})(5s^{1/4} - 1). \tag{7}$$

The limiting value of $s^{1/4} = 1/5$ ensures non-negative α^2. For values of s below this limiting value, the paraboloid floats with its base above the water surface, taking configurations A or B depending on the values of α. For the values of s and α lying in the region bounded by B–C curve and the $s^{1/4}$-axis, the base of the paraboloid submerges in water. The critical values of α for transitions A to B ($\alpha = 0.89$) and B to C ($\alpha = 0.73$) are marked in the figure for the specific case of $s^{1/4} = 7/11$. The maximum tilt angle in this case is 45° as can be verified from equation (5). The

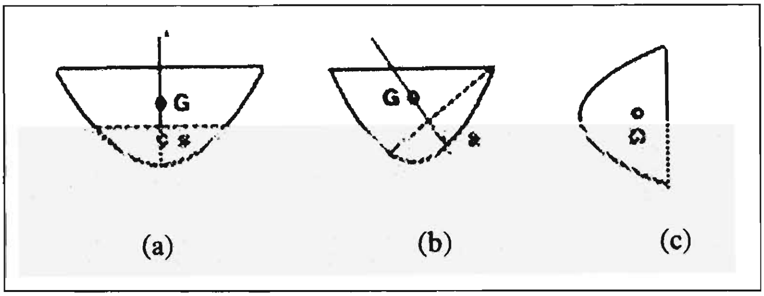


Figure 6. Tumbling of a Paraboloid. $s^{1/4} = 7/11$.
(a) $m = 0$; $\alpha = 0.89$; Transition from Configuration A to B.
(b) $\alpha = 0.72$; $m = 1$; Transition from Configuration B to C.
(c) $\alpha = 0.6$; Configuration C.

corresponding configurations at these critical values are shown in *Figure* 6. The A-map in *Figure* 5 contains a smaller ellipse corresponding to configuration C inside a larger ellipse corresponding to the A–B transition. The region between the two transitions does not exist for the cylinder shown in *Figure* 2. The tilted configuration is a unique feature for the paraboloid, and this feature persists for a general *polyboloid* given by $z^n = k\,(x^2+y^2)$, for $n \geq 1$.

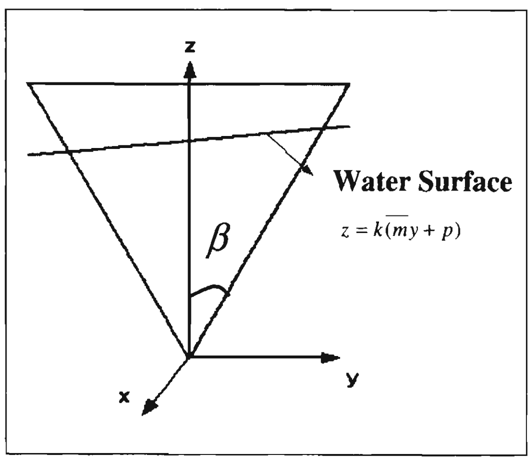
6. A-Maps: Cone, Parabolic Prism

The inclined configuration analysis for the right circular cone of base radius R and height H is done in a similar manner as described in the previous section. The equation of the cone is given by

$$z^2 = k^2(x^2 + y^2) \tag{8}$$

The z-axis of the Cartesian coordinates is taken along the axis of the cone as shown in *Figure* 7. Here, $k = H/R = 1/\alpha = 1/\tan\beta$ where β is the semi-apex angle of the cone. In the vertical configuration, the height of the immersed volume is given by $s^{1/3} = h/H$. For the tilted configuration, the immersed volume V_{IM} and the coordinates of the buoyancy centre B are given by

Figure 7. Schematic of tilted cone.



$$V_{IM} = \iiint\limits_{V_{IM}} dV = \frac{\pi}{3} k^4 \bar{p}^3 \frac{(1+m^2)^{1/2}}{(k^2 - m^2)^{3/2}};$$

$$B_x = 0; B_y = \frac{1}{V_{IM}} \iiint_{V_{IM}} y dV = \frac{3}{4} \frac{k^2 \overline{pm}}{k^2 - m^2};$$

$$B_z = \frac{1}{V_{IM}} \iiint_{V_{IM}} z dV = \frac{3}{4} \frac{k^3 \overline{p}}{k^2 - m^2}. \tag{9}$$

Here $\bar{p}$ is obtained, using the condition of the constancy of the immersed volume, to be

$$\bar{p}\, k = (k^2 - m^2) H / (1 + k^2). \tag{10}$$

The condition for the line BG (where B is the centre of buoyancy and G is the centre of gravity) to be perpendicular to the water surface can be obtained as

$$\frac{(1 + k^2)^6}{k^6} s^2 = (k^2 - m^2)^3 (1 + m^2). \tag{11}$$

For a vertical configuration this equation reduces to

$$\alpha^2 = \frac{1 - s^{1/3}}{s^{1/3}} \tag{12}$$

as shown by Bansal [2]. Stable vertical configurations of a cone for a given value of s are possible only for the values of the aspect ratio α given by equation (12).

A similar analysis to that of a paraboloid can reveal that the maximum angle of inclination for a stable, tilted cone at a given value of s reduces to a simple equation $m = \alpha$.

Figure 8 shows the A-Map of the solid cone. Equation (12) is plotted as a solid line which marks the transition from Configuration A to B. Here, the semi-apex angle β of the cone is taken to be the abscissa and s is taken as the ordinate. The dotted line is plotted using equation (11) which shows the transition from Configuration B to C. It can be noted that there is no non-zero lower bound on s for configuration C unlike in the case of a paraboloid.

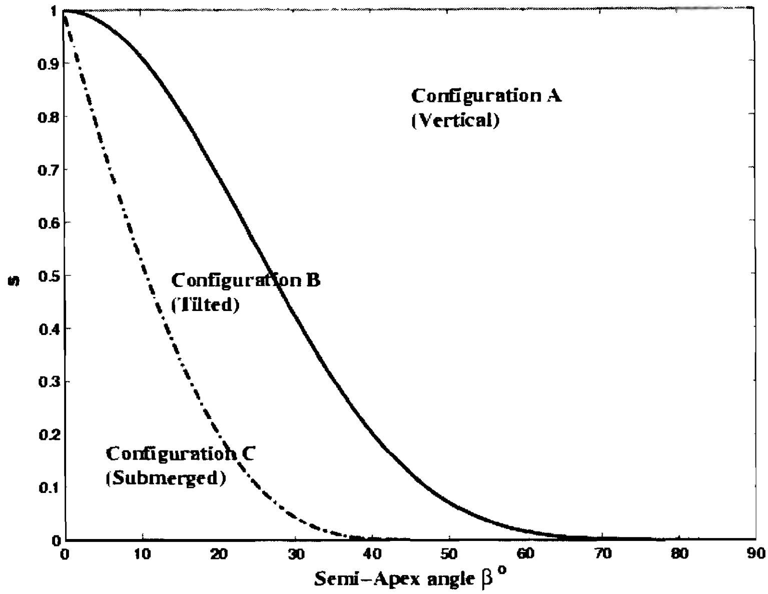


Figure 8. A-maps of a right circular cone.

The A-map for a long parabolic prism (of infinite length!) closely resembles that of the paraboloid. A similar analysis for the parabolic prism yields the condition for a stable tilted configuration as

$$\frac{12}{5\alpha^2}(1-s^{2/3}) = 2+m^2 \tag{13}$$

The corresponding equations for transitions from configuration A to B and B to C are

$$\frac{12}{5\alpha^2}(1-s^{2/3}) = 2 \tag{14}$$

and

$$\alpha^2 = \frac{4}{5}(1-s^{1/3})(4s^{1/3}-1) \tag{15}$$

respectively. Similar to the case of the paraboloid, the limiting value of $s^{1/3} = 1/4$ ensures non-negative α^2. Below this value of s, the prism floats with its base above the water surface, taking configurations A or B depending on the values of α.

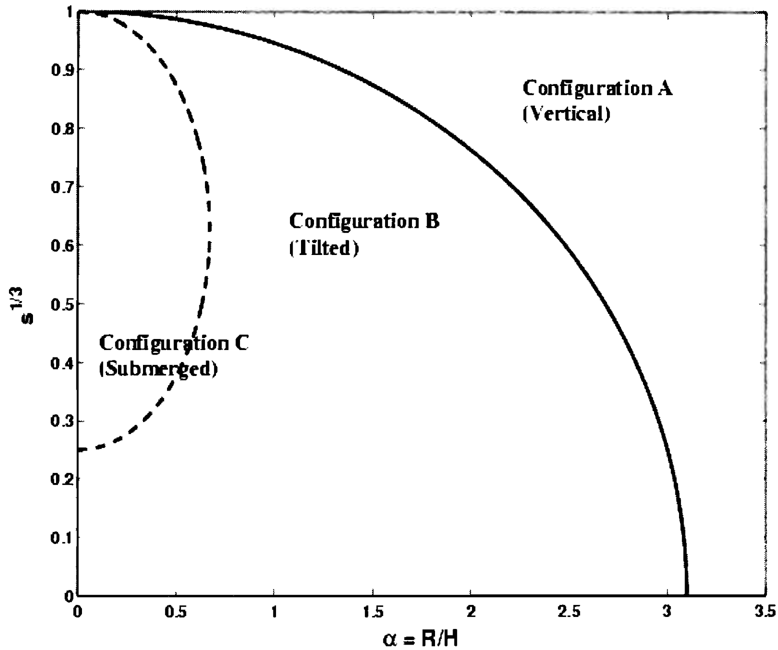


Figure 9. A-Maps of a parabolic prism.

7. Check for Stability of Configuration A.

The Archimedean criterion for the case of $m=0$ for a paraboloid is given by equation (6). This criterion can be derived independently for small disturbances from the stable configuration leading to the *metacentre* concept. Accordingly, as shown in hydrostatic texts, the distance between the centre of buoyancy and the metacentre is given by $\mathrm{BM}=I/V_{IM}$, where I is the moment of inertia of the cross-section of the floating body *at* the water surface, V_{IM} is the immersed volume below the water surface. It can be easily seen that the distance BG for a right circular cone floating vertically is $3(H-h)/4=3H(1-s^{1/3})/4$. The moment of inertia of the cross-section of the body at the water surface is $\pi r^4/4$ and the immersed volume is given by $V_{IM}= \pi r^2 h/3$ giving $\mathrm{BM}=(3/4).(h/k^2)$. For stable equilibrium, $\mathrm{GM} \geq 0$. That is, $\mathrm{GM}=\mathrm{BM}-\mathrm{BG} = (3/4)s^{1/3}H\alpha^2-(3/4)H(1-s^{1/3}) \geq 0$, which, in the limit of $\mathrm{GM} = 0$, yields equation (12). Therefore, we have shown here that for a stable vertical configuration of a right circular cone, the metacentre M coincides with G. This result can be proved for a right paraboloid too.

8. Tumbling Paraboloidal Iceberg

Similar to a cylinder tumbling when it exceeds a critical height, a paraboloid tumbles when the base angle exceeds a critical value. Rorres [3] cites Jules Verne in *20,000 Leagues Under the Sea*: *"An enormous block of ice; a mountain turned over. When icebergs are undermined by warmer waters or by repeated collisions, their center of gravity rises, with the result that they overturn completely"*.

Rorres illustrates the idea with a paraboloidal iceberg (specific gravity=0.9). The iceberg is in vertical equilibrium (configuration A) for base angles < 82.54°. As the iceberg melts the tilt angle increases from 0° to 12.3°, when the second transition occurs for a base angle of 82.65°. These angles can be predicted from the curves shown in *Figure* 4.

This example is re-illustrated taking a smaller value for the specific gravity such that $s^{1/4}=7/11$ It can be shown that the critical angle of tilt increases to 45° in this case. The corresponding configurations are shown in *Figure* 6.

9. Closure

It is indeed remarkable that Archimedes posed and solved a hydrostatic problem of eternal interest to students, teachers and naval architects. In this article, we extended the Archimedes analysis to the case of a right circular solid cone. The cone behavior is similar to that of a paraboloid.

Finally, we discussed a two-dimensional case of a long parabolic cylinder. In all the cases we saw that the limiting case of configuration A can be derived using the formula $BM=I/V_{IM}$. As commented by Rorres, Archimedean concepts and results have deep significance for mathematical analysis. Rorres also provides valuable references (numbered 7, 9, 11 in [3]) that deal with other polyhedral bodies. These references cite other classic texts on hydrostatics by eminent mathematicians and teachers like Lamb, Loney and Ramsey who shaped engineering education in the early part of the 20th century. Bowing to market forces we are

It is indeed remarkable that Archimedes posed and solved a hydrostatic problem of eternal interest to students, teachers and naval architects.

Exercises

1. Repeat the analysis for a long isosceles triangular prism.

2. Find the transition angles for a solid cone for specific gravity $s = 0.5$.

3. Find the specific gravity of a parabolic cylinder the critical angle for B–C transition is 30^{o}.

4. Prove the results for immersed volumes and the coordinates of the buoyancy centres given in equations (2) and (9). Using these results, show that the immersed volume enclosed between the plane $z=2y+3$ and the paraboloid $z=x^2+y^2$ is 8π. The corresponding coordinates of B are (0,1,11/3).

5. Prove that the immersed volume for a tilted parabolic cylinder given by $z=ky^2$ is $k/6\,(\overline{m}^{\,2}+4\overline{p}^{\,2})^{3/2}$ where the water surface is given by $z=k(\overline{m}y+\overline{p})$.

6. Prove that the A–B transition for a general polyboloid is given by the equation $u^{1-n}-u^2=(n+2)\alpha^2/4$, where $u=s^{1/(n+1)}$, $n\geq 1$.

disregarding teaching educational history and heritage in schools and colleges. Too much emphasis on technology-driven science (TDS) may not be sustainable or viable due to volatile trends in technology. Ancient technology drew inspiration from science, but today science seems to be blinded by technology.

10. Epilogue

Recently, a Norwegian ship *Cuebride* carrying 80MN (8000 tonnes) of LPG got stuck in soft mud at Ratnagiri port (Deccan Herald, Sept. 18, 2006). Two tugs were deployed to pull the ship out of the mud *safely* without the ship tumbling. Perhaps Archimedes would have come up with his own idea to salvage the ship!

Suggested Reading

[1] SW Yuan, *Foundations of Fluid Mechanics*, Prentice Hall of India, 1969.
[2] R K Bansal, *Fluid Mechanics and Hydraulic Machines*, Laxmi Publications Ltd., October, 2000.
[3] C Rorres, *The Mathematical Intelligencer*, Vol.26, No.3, pp.32–42, 2004.