Fluid Statics and Archimedes

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The equations of fluid statics derive from the fact that in a stationary fluid only normal stress (pressure) is present and tangential stresses are absent. In this article we derive equations for variation of pressure in a static fluid or in one that is moving as a rigid body. Also, we show that surfaces of constant pressure and constant density should coincide. Various types of instabilities related to submerged bodies and multi layerd fluids are briefly discussed.

Pressure in Fluids

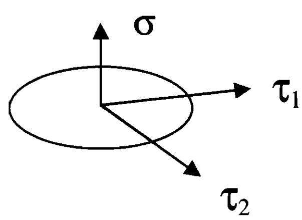
A body that is partly or fully surrounded by a fluid experiences a force, especially when the fluid happens to be a liquid like water. Presumably, the feeling of lightness of his body in the bathtub enlightened Archimedes towards his famous buoyancy principle. A shower instead of a bathtub may have perhaps delayed the discovery of the principle by a few centuries! Archimedes formulated his laws on buoyancy and floating bodies in about 250 BC. What is remarkable is that these laws were formulated more than 1800 years before Galileo and Newton made the next major advances in mechanics.

Any surface immersed in a fluid is subject to pressure. In fact, due to the air above us, our bodies are subject to a pressure of about 1 kg/cm^2, i.e., about half-a-ton of force on our head (yet, we don't feel it, why?). In a stationary fluid or in one that is moving as a rigid body, there are only two types of forces: one due to pressure and the other due to body forces, like from gravity and centrifugal acceleration. Pressure is a subset of what is called stress in the study of mechanics of solids and fluids (see *Box*1). Just as temperature and density, stress can be given at each and every point within a substance[1]. These quantities may vary continuously in space and time. Thus we can talk about a

Box 1.

To talk about stress we need an area, either real or imaginary.

Stress is force per unit area. Since both force and area are vectors, stress becomes a tensor. A vector requires 3 components to define it (three force components in three reference directions) whereas a tensor requires 9 components. On a given elemental area, force componen perpendicular or normal to the area produces normal stress (σ); the two force components in two orthogonal directions tangential to the area produce two shear stresses (τ_1,τ_2).



temperature field, a pressure field or a stress field.

In fluids, there are two important points to remember about pressure:

1. Pressure is always normal to the surface and by definition it is positive when it is into the surface. Thus it is a 'normal stress' and is positive when it is compressive or trying to squeeze.

2. Pressure at a point is equal in all directions.

These two characteristics about pressure are retained even when a fluid is moving. In a stationary fluid stress consists of only pressure. *There are no tangential or shear stresses in a stationary fluid;* even a minute shear stress will cause fluid to move, more correctly deform continuously. A fluid is often defined as one that cannot sustain a shear stress in a stationary fluid. There is no 'friction' and the pressure field is independent of viscosity.

[1] In continum mechanics, which includes fluid and solid mechanics, it is assumed that parameters like temperature, velocity and density are continuous. At molecular scales, clearly, these quantities cannot be defined. But often we are interested in scales much larger than intermolecular distances and when averaged over several thousand molecules, quantities like temperature become 'continuous'. Effects of molecular motions, however, need to be considered and show up as diffusion of heat and viscous stresses in the governing equations.

What Causes Pressure and How Does it Vary Within a Fluid?

In fluid that is enclosed on all sides, as in a balloon or a gas cylinder, pressure is applied by the walls and transmitted to all points within the fluid. The pressure level is of course determined by the initial filling process. That this pressure is transmitted without loss, is the famous Pascal's law. Compressing the fluid (reducing the volume) would clearly increase the pressure.

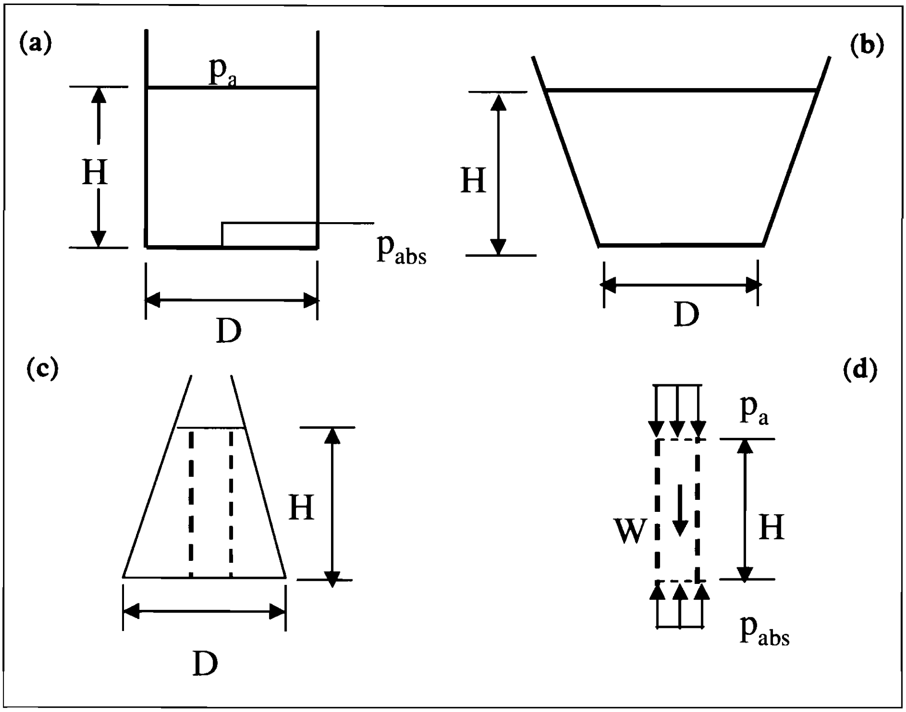


Figure 1. Pressure on the bottom surface is same in all the three containers. Equilibrium of vertical slice (d) shows that $p_{abs}=p_a+\rho gH$; note that shear stress is zero.

We will see below that within a stationary fluid pressure varies due to a body force, often the weight. In the atmosphere the top is 'open' and pressure is due to the gravitational body force or weight of the fluid. To support the weight above, pressure increases with depth.

Consider a liquid of density ρ in a container with vertical sides walls, horizontal bottom wall and cross-sectional area A (*Figure* 1a) Equilibrium of the liquid in the vertical direction gives

$$p_{abs} A - p_a A = \text{weight of liquid} = \rho\, g\, H\, A,$$

where p_{abs} is the pressure on the bottom wall, p_a is atmospheric pressure and H is liquid height.

$$p_{abs} = p_a + \rho\, g\, H.$$

(It is often convenient to subtract the atmospheric pressure from the total or absolute pressure to get gauge pressure, $p = p_{abs} - p_a$.)

In the above relation we have used the fact that the *side walls do not apply any shear stress on the liquid*. For example, sand instead of a liquid in the container will give a lower average pressure on the bottom wall because part of the weight is supported by friction on the side walls. We have also assumed that the pressure is constant on the bottom surface (which is horizontal) – we will show below that pressure is uniform on a horizontal surface.

Sand instead of a liquid in the container will give a lower average pressure on the bottom wall.

The pressures, on the bottom of containers that have non-vertical side walls (*Figures* 1b and 1c) will be the same, ρgH, even though the weights of the liquids in the three containers are different. We can easily show this result from the vertical equilibrium of a vertical slice fluid (*Figure* 1d) of height H and again using the fact that shear stresses are absent.

In general, difference in the pressures at points 1 and 2 located at height difference h, is

$$\Delta p_1 = p_2 - p_1 = \rho g h . \quad (1)$$

A more general relation for the vertical variation of pressure (see *Box* 2) is

$$\partial p / \partial z = - \rho g, \quad (2)$$

where z is vertical coordinate increasing upward.

Integrating

$$p = - \int \rho g \, dz + p_0 , \quad (3)$$

where p_0 is the pressure at $z = 0$.

If ρ and g are constants, p reduces linearly with height,

$$p = - \rho g z + p_0$$

and we recover (1).

Similarly, equilibrium in the horizontal (x, y) directions gives

$$\partial p / \partial x = 0, \quad \partial p / \partial y = 0;$$

pressure is constant in any horizontal plane.

Box 2.

General relations may be derived for variation of pressure in a stationary fluid.

Take an elemental fluid volume of sides Δx, Δy and Δz with pressure p at the centre of the volume. (See *figure* below) Equation of equilibrium in the vertical direction gives

$$\Delta p\,(\Delta x\,\Delta y) = -\rho\, g\, \Delta x\, \Delta y\, \Delta z \qquad \text{(i)}$$

$$\Delta p = -\rho\, g\, \Delta z$$

and in the limit $\Delta z \to 0$,

$$\partial p / \partial z = -\rho g. \qquad \text{(ii)}$$

Integrating

$$p = -\int \rho g\, dz + p_0 \qquad \text{(iii)}$$

where p_0 is the pressure at $z = 0$.

If ρ and g are constants, p reduces linearly with height and

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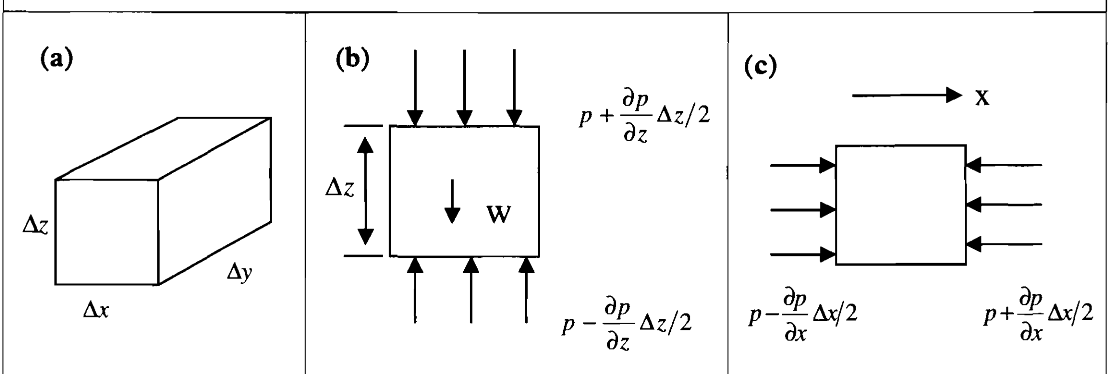
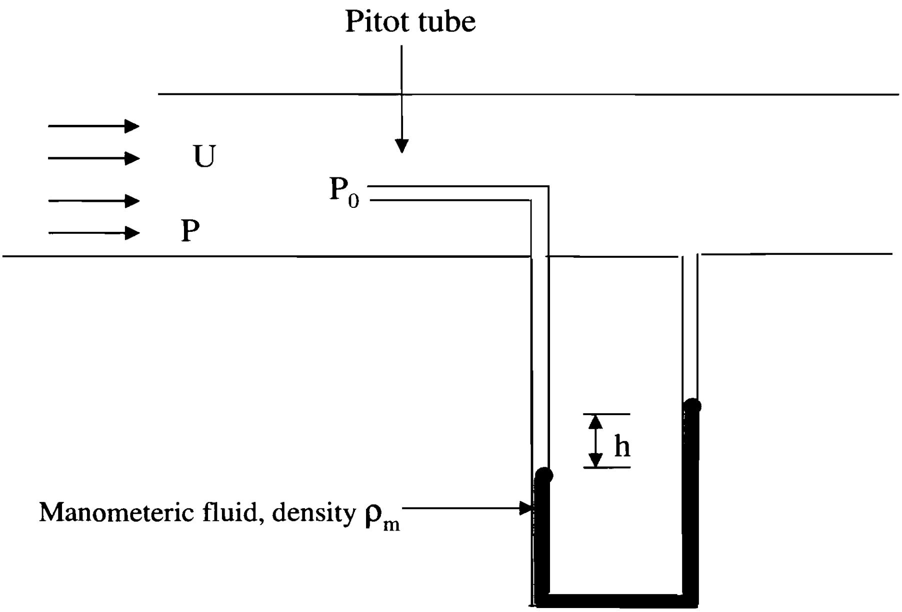


Equilibrium of an elemental volume (a) in the vertical direction (b) and in the horizontal direction (c). Pressure increases with depth to support the weight and does not vary in the horizontal direction.

Box 3. Fluid Velocity Measurement using a Pitot Tube.



Pressure differences can be measured using a U tube manometer. A pitot tube facing a flow experiences pressure $p_0 = p + 1/2\ \rho_f\ U^2$ and creates a deflection h; $\rho_m gh = p_0 - p = 1/2\ \rho_f\ U^2$. p is the pressure in the flow (so called static pressure) and p_0 is the stagnation pressure. The fluids in the pitot tube and manometer tube are stationary. In the above expression for h the flowing fluid is assumed to be a gas with density (ρ_f) $\ll$ manometric fluid density ρ_m. If ρ_f is not negligible then pressure change in the manometer due to the fluid has to be included.

The fact that pressure is transmitted without friction is used in hydraulic and pneumatic presures and in measurement of pressure, for example, using U-tube manometers (see *Box* 3).

We have seen how pressure varies within a static fluid, but is there any restriction on how density can vary. We know that the interface between two immiscible liquids, e.g., oil on water, is horizontal (*Figure* 2). In each layer pressure will increase linearly, proportional to the local density.

The result that layers of constant density are horizontal is more general. Density can vary continuously due to, say, temperature variation as in the atmosphere, or variation in salt concentration as in oceans. In such cases it can be shown that *planes of constant*

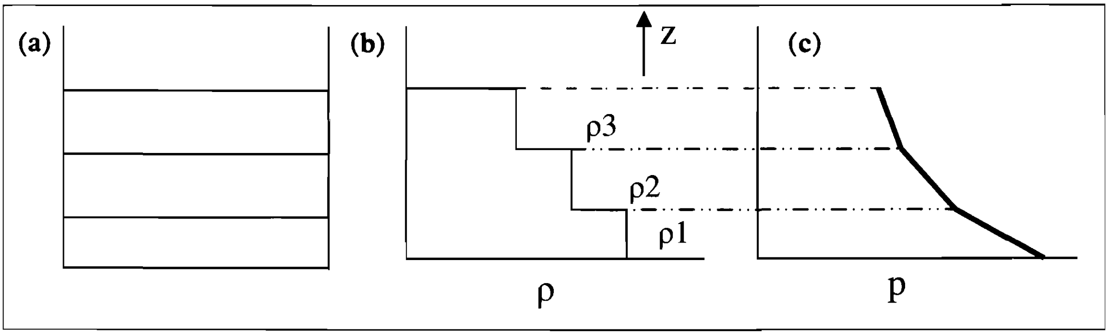


Figure 2. (a) Three layers of immiscible liquids. (b) The variation of density with height. For equilibrium, the interfaces between the liquids have to be horizontal. From stability considerations the heaviest liquid is at the bottom and the lightest at the top. (c) The pressure increases linearly in proportion to the density in each layer.

density are horizontal; this in turn implies that the variable that is causing the density variation (e.g. temperature) is constant on a horizontal plane. Of course, this result is true when there is no motion, i.e., the fluid is static. Any lateral variation in density will result in horizontal pressure difference and an immediate motion to remove this lateral variation. The pressure variation in the vertical direction is governed by equations 2 and 3.

Thus we come to the main results of hydrostatics:

1. In the pressence of gravity pressure varies linearly with depth for a fluid with constant density. Use equation (3) if density is not constant.

2. Pressure is constant along any horizontal plane.

3. Density can vary only in the vertical direction and is constant on any horizontal plane.

Fluid Accelerating as a Rigid Body

As for a stationary fluid, shear stresses are also zero when fluid is moving as a rigid body. The results for pressure variation are easily extended when fluid is linearly accelerating as a 'rigid' body with acceleration components a_x, a_y and a_z

$$\partial p / \partial z = -\rho g - \rho a_z,\ \partial p / \partial x = -\rho a_x,\ \partial p / \partial y = -\rho a_y,$$

where z axis is vertically upwards and x and y are in horizontal directions.

Surfaces of constant pressure and surfaces of constant density are both horizontal.

For example, a closed box of air accelerating in the x direction will have, assuming density is constant, the pressure field

$$p = -\rho g z - \rho a_x x + p_0.$$

Acceleration in a horizontal direction (*Figure* 3a) causes the free surface of a liquid in a container to tilt; surfaces of constant pressure and, in cases of stratified fluid, surfaces of constant density tilt by same amount.

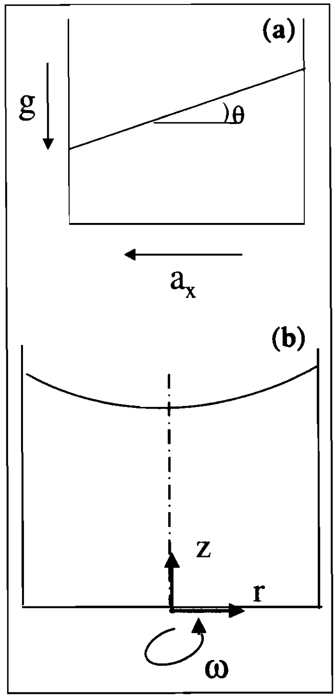


Figure 3. (a) Top. Free surface tilts when the container accelerates in the horizontal direction tan θ =a_x/g. (b) Bottom. A liquid in a rotating container rotates as a rigid body. The free surface is a paraboloid of revolution.

A frequently occurring situation is a fluid rotating as a rigid body. The pressure increases radially to support the centripetal acceleration ($\rho\, \omega^2 r$) of the fluid particles,

$$\partial p / \partial r = \rho \omega^2 r$$

and $$p = -\rho g z + (\rho\, \omega^2 r^2 / 2) + p_0.$$

Here ω is angular velocity and r is radial distance.

Region of fluid having rigid body rotation is called a forced vortex. Pressure varies quadratically with r; the free surface is a paraboloid of revolution (*Figure* 3b). This result is used to cast large parabolic mirrors.

In a so called free vortex, the tangential velocity varies inversely with radius, $V_t = C / r$, and $p = -\rho g z + \rho C^2/r^2$.

Often the 'vortices' in nature like whirlpools have rigid body rotation near the centre, the so called core of the vortex. Outside the core, the flow corresponds to a free vortex. The centre or the eye of the vortex is a point of minimum pressure.

Buoyancy and Archimedes Principle

Every high-school student is taught the Archimedes principle of buoyancy: A body partially or fully submerged in a fluid experiences an upward (buoyancy) force F_B equal to the weight of the fluid displaced.

$$F_B = \rho V g,$$

where ρ is fluid density, V is volume of fluid displaced and g is

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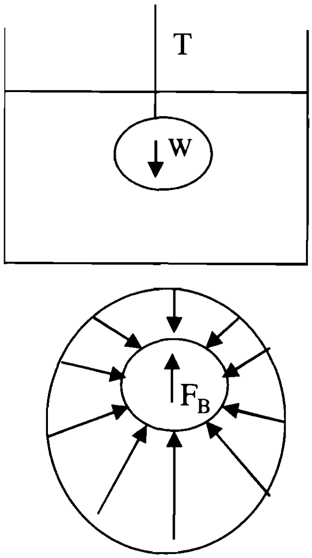

Figure 4. A body immersed in a fluid feels an upward force F_B due to the pressure increasing with depth. There is no horizontal force, pressure acts normal to the surface at each point. The tension in a string supporting a body is $T = W - F_B$.

acceleration due to gravity. The increasing pressure with depth causes the buoyancy force (*Figure* 4).

One of the best ways to understand buoyancy force is in the following way: Imagine the body is replaced by a parcel of the same fluid having the *same shape* and *same volume* as the body. This fluid volume will be in equilibrium. An upward force due to fluid pressure must exist which balances the weight of the fluid parcel. The pressure field is the same in both cases i.e., when the solid body is present and when it is replaced by the imagi-nary fluid. Thus the body will feel the same upward force = ρVg.

Archimedes principle is easily derived for a vertical cylinder with cross-sectional area A and length L (*Figure* 5a).The pressure difference between the two sides, from equation (1), ρgL multiplied by A gives the buoyancy force = $\rho gLA = \rho gV$. For an arbitrary shaped body we may integrate over a large number of 'cylinders' with elemental area dA to get Archimedes principle (*Figure* 5b),

$$F_B = \int \rho\, g\, L\, dA = \rho\, g\, V.$$

What happens if a body is submerged in a multilayered fluid or in a stratified fluid? The above line of thinking will immediately lead us to conclude that the Archimedes principle is exactly the same; i.e. buoyancy force will be the weight of the fluid displaced.

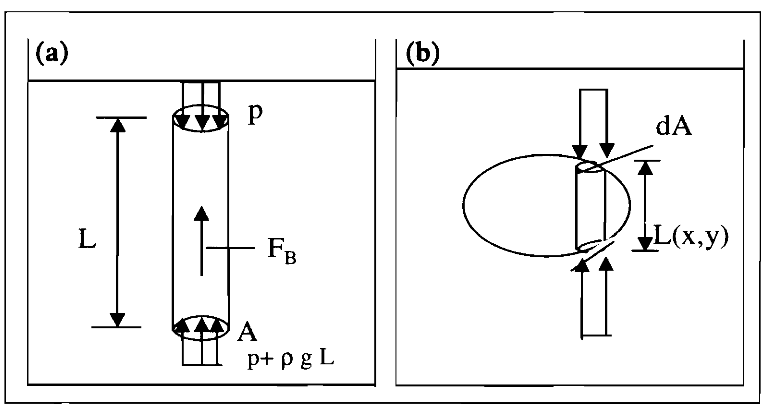

Figure 5.
a) Derivation of Archimedes principle for a submerged vertical cylinder $F_B = \rho\ gLA = \rho gV$. b) For an arbitrary body integration over 'elemental' cylinders gives the Archimedes buoyancy force.

In fact, a more general Archimedes principle may be derived for a body immersed in a fluid that is accelerating as a rigid body. Again replacing the body with the fluid gives the forces due to the fluid pressure. For example, a body placed in a fluid accelerating in the vertically upward direction with acceleration a_z will feel a 'buoyancy' force $F_{Bz} = \rho(g+a_z)\,V$; an acceleration a_x in the horizontal direction will give a 'buoyancy' force $F_{Bx} = \rho a_x V$ in the direction of the acceleration. The pressure differences accelerating the fluid also act on the submerged body to produce these forces.

Archimedes principle also holds in multilayered or stratified fluids.

Questions of Instability

For a stable state to exist, a system has to be not only in equilibrium but in stable equilibrium. A pencil standing exactly vertical on its pointed end can remain so if there is absolutely no disturbance. A slightest wind will topple it. Similarly a heavier fluid (say water) can lie on a lighter one (air). We never observe it though, because such a configuration is not stable. A small perturbation of the interface will grow indefinitely till the water and air exchange places (see *Figure* 6). We do see, however, water lying over air in narrow tubes. How do we explain this? How narrow should the tube be? (The reader can find the answers to these questions.) The instability of heavier fluid over a lighter one is termed Rayleigh–Taylor instability and mixing of the two fluids in such a system is a fundamental and intense field of research.

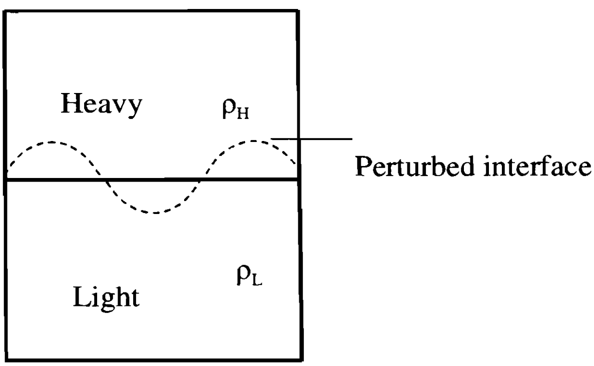

Figure 6. A heavier fluid over a lighter fluid can be in equilibrium, but it is not stable. A small perturbation of the interface (shown in dotted line) will grow and the heavier fluid finally settles below the lighter one.

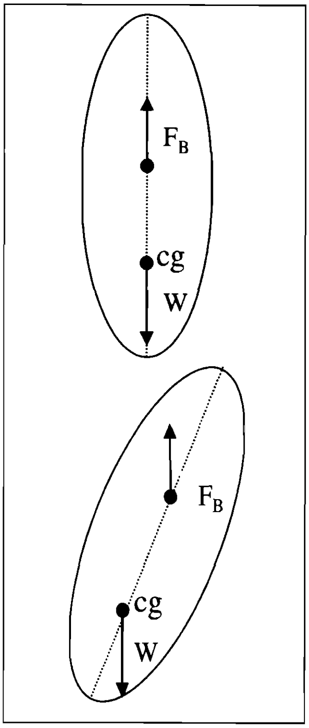

Figure 7. For stability of a fully submerged body the centre of gravity (cg) should be below the centre of buoyancy. Any tilting will give a righting movement.

Density increasing with height is termed unstable stratification and as observed in oceans and in the atmosphere, leads to immediate overturning and mixing. Unstable stratification, for example, can be caused by the heating of air near the ground by the sun, release of latent heat by condensation of water vapour in clouds or in everyday experience when we heat water on a stove. In all these cases we obtain vigorous convection – rising of lighter hotter fluid and falling of heavier cooler fluid. In cases of stable stratification, with density reducing with height, mixing is difficult. Inversion layers in the atmosphere are stably stratified and exhaust emissions and other pollutants remain near the ground causing great discomfort. Fluid motion caused entirely by buoyancy is termed natural or free convection. Another common occurrence of natural convection is fluid rising from a hot source like a candle or an agarabathi.

There are other instabilities with regard to buoyancy. Archimedes made a thorough study of stability of submerged bodies. For a fully submerged body like a fish or submarine to be stable the centre of gravity should lie below the centre of buoyancy (*Figure* 7); centre of buoyancy is the centre of gravity of the displaced fluid.

Stability of a semi–submerged body like a ship is more complicated and more interesting, and its analysis of clearly practical value. (See article in Classroom section of the present issue.) Here the centre of gravity can and often does lie over the centre of buoyancy but the body can still be stable. The key to understanding stability of semi-submerged bodies lies in realizing *tilting of a body also shifts the centre of buoyancy* in relation to the

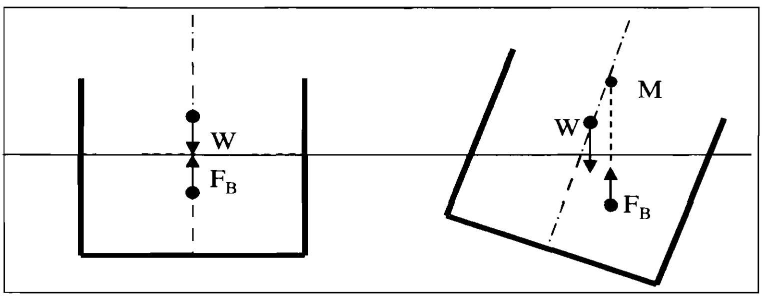

Figure 8. A semi-submerged body like a boat can be stable even if the centre of gravity of the body lies above the centre of buoyancy. When tilted the centre of buoyancy shifts to give a stabilizing moment. For stability the metacentre M should lie above the cg. Metacentre is the point where the vertical line passing through the centre of buoyancy intersects the line of symmetry.

body and provides a righting moment (*Figure* 8). (In a fully submerged body centre of gravity and centre of buoyancy don't shift positions with respect to the body as the orientation changes.) The metacentre height determines whether a semi-submerged body is stable or not (see *Figure* 8). Metacentre lying above the centre of gravity ensures stability. Stability is entirely determined by the vertical location of the centre of gravity and geometry of the cross-section of the body cutting the free surface. A lower centre of gravity and wider cross-section both result in greater stability. Of course, in all cases of equilibrium of submerged and semi-submerged bodies, buoyancy force must be equal to the body weight (force equilibrium) and the two must lie in the same vertical line (moment equilibrium).

There are other instabilities, important for balloons, airships, submarines and in the atmosphere. Imagine a balloon whose volume can change as it rises or falls. The weight of the balloon material and the air inside is constant. The buoyancy force, however, can change as the volume of the balloon and density of the air outside change. The volume of the balloon can change depending on the temperature and pressure of the gas inside. The temperature in turn changes due to heat transfer to the ambient and because of sun's radiation. Some balloons like hot air balloons have to be analysed differently because they have openings at the bottom through which gas can flow in and out;in addition, the balloons can be limp (volume can change) or taut.

Similar arguments are used to determine stability in the atmosphere. The earlier reasoning of unstable stratification may be modified by these more complex interactions. Both moisture and temperature play a role in atmospheric stability, as do salinity and temperature in the oceans. In fact the different rates of diffusion of temperature and salt lead to interesting and sometimes non-intuitive forms of instability.

The basis for all stability analysis is the same: perturb the system from equilibrium and check if the perturbation grows (usually with time).

Suggested Reading

[1] L Prandtl and O G Tietjens, *Fundamentals of Hydro-and Aero-Mechanics*, Dover Publications, 1957.

[2] R K Bansal, *Fluid Mechanics and Hydraulic Machines*, Laxmi Publications Ltd., Oct. 2000.

[3] Jaywant H Arakeri and P N Shankar, Ludwig Prandtl and Boundary Layers in Fluid, *Resonance*, December, 2000.

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